

# Symplectic Lefschetz fibrations with arbitrary fundamental groups

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(with an appendix by Ivan Smith)

## Abstract

In this paper we give an explicit construction of a symplectic Lefschetz fibration whose total space is a smooth compact four dimensional manifold with a prescribed fundamental group. We also study the numerical properties of the sections in symplectic Lefschetz fibrations and their relation to the structure of the monodromy group.

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# 1 Introduction

In this paper we give an explicit construction of a symplectic Lefschetz fibration whose total space is a smooth compact four dimensional manifold with a prescribed fundamental group. The existence of such a fibration is also a consequence of the remarkable recent work of Donaldson [Donaldson98] (see also [Auroux98]) who proved the existence of a Lefschetz pencil structure on any symplectic 4-manifold and the results of Gompf [Gompf95], who proved that any finitely presentable group can be realized as the fundamental group of a symplectic 4-manifold.

Since Donaldson's proof is non-constructive, as an alternative we present a direct purely topological construction of symplectic Lefschetz fibrations which is effective and allows an explicit control on the number of singular fibers. The construction is based on an algebraic geometric method for creating positive relations among right handed Dehn twists. The ubiquity of such relations combined with a simple group theoretic characterization of symplectic Lefschetz fibrations due to Gompf (see Proposition 2.3) turns out to be sufficient for the construction.

Before we can state our main theorem we need to introduce some notation. For any integer  $n \geq 0$  denote by  $\pi_n$  the fundamental group of a compact Riemann surface of genus  $n$ . As usual a group  $G$  is called *finitely presentable* if it can be written as a quotient of a free group on finitely many generators by a subgroup generated by the conjugacy classes of finitely many elements. By a *finite presentation* of a group  $G$  we mean a surjective homomorphism  $A \twoheadrightarrow G$  from some finitely presentable group  $A$  onto  $G$  so that  $\ker[A \rightarrow G]$  is generated as a normal subgroup by finitely many elements in  $A$ .

**Theorem A** *Let  $\Gamma$  be a finitely presentable group with a given finite presentation  $a : \pi_g \twoheadrightarrow \Gamma$ . Then there exists a surjective homomorphism  $b : \pi_h \rightarrow \pi_g$  for some  $h \geq g$  and a symplectic Lefschetz fibration  $f : X \rightarrow S^2$  such that*

- (i) *the regular fiber of  $f$  is of genus  $h$ ,*
- (ii)  $\pi_1(X) \cong \Gamma$ ,
- (iii) *the natural surjection of the fundamental group of the fiber of  $f$  onto the fundamental group of  $X$  coincides with  $a \circ b$ .*

Note that any finitely presentable group  $\Gamma$  admits a finite presentation of the form  $\pi_g \twoheadrightarrow \Gamma$  since  $\pi_g$  surjects onto a free group on  $g$  generators.

The map  $b$  in the above theorem is not arbitrary. It factors as

$$\begin{array}{ccc} \pi_h & \xrightarrow{b} & \pi_g \\ & \searrow & \nearrow \\ & \pi_e & \end{array}$$

where the surface of genus  $e$  is obtained from the surface of genus  $g$  by adding handles and the surface of genus  $h$  is obtained from the surface of genus  $e$  as a ramified finite covering.

Our second theorem concerns symplectic fibrations of Lefschetz type over curves of higher genus.

**Theorem B** *Let  $\Gamma$  be a finitely presentable group with a given presentation  $a : \pi_g \twoheadrightarrow \Gamma$ . Then there exist a surjective homomorphism  $\pi_e \rightarrow \pi_g$  and a symplectic Lefschetz fibration  $f : X \rightarrow C_k$  over a smooth surface  $C_k$  of genus  $k$  so that*

- (i) *the regular fiber of  $f$  is of genus  $e$ ,*
- (ii)  *$f$  has a unique singular fiber*
- (iii) *the fundamental group of  $X$  fits in a short exact sequence*

$$1 \rightarrow \Gamma \rightarrow \pi_1(X) \rightarrow \pi_k \rightarrow 1,$$

- (iv) *the natural surjective map from the fundamental group of the fiber of  $f$  to  $\Gamma$  coincides with the composition  $\pi_e \rightarrow \pi_g \xrightarrow{a} \Gamma$ .*

This work is an elaboration on a discussion at the end of [Bogomolov-Katzarkov98]. We describe in details an enhancement of the general technique for constructing examples of symplectic fibrations used in [Bogomolov-Katzarkov98]. Our proof is based on exploiting the correspondence between subgroups of the mapping class group and graphs of vanishing cycles.

In general it is expected that every smooth four-dimensional manifold is diffeomorphic to an achiral Lefschetz fibration possibly after some stabilization. The purpose of this paper is to study what other conditions besides chirality determine the SLF among all fibrations.

The paper is organized as follows. In section two we give some preliminaries on subgroups of mapping class groups generated by Dehn twists and recall an important result of Gompf which characterizes symplectic Lefschetz fibrations via their monodromy representations. In section three we explain the general construction and prove Theorems A and B. In section four we give an explicit example of a symplectic Lefschetz fibration of genus three Riemann surfaces whose total space has first Betti number one and a different construction of a Lefschetz fibration whose total space has a fundamental group isomorphic  $\mathbb{Z}$ . All this demonstrates the flexibility of the construction for obtaining interesting examples.

The second construction is similar in spirit to a computation done by Donaldson in which he has represented Thurston's example as a symplectic Lefschetz fibration of genus three Riemann surfaces. A whole series of examples of the same flavor was constructed independently in [Ozbagci-Stipsicz98] and [Smitha, Smithb].

In the last section we apply the group theoretic part of the construction to the study of the numerical properties of sections in symplectic Lefschetz fibrations. Finally Appendix A, written by Ivan Smith, presents a short proof of the non-existence of SLF with monodromy contained in the Torelli group.

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## Notation and terminology

$C_g$  a smooth compact oriented surface of genus  $g$ .

$\Delta$  an analytic disk.

$\Gamma$  a finitely presentable group.

$\text{Map}_g(-)$  the subsemigroup in  $\text{Map}_g$  generated by the right Dehn twists  $t_r, r \in R$ .

$\text{Map}_g^n$  the mapping class group of a smooth genus  $g$  surface with  $n$  punctures.

$\text{Map}_g r^n$  the mapping class group of a smooth genus  $g$  surface with  $n$  punctures and  $r$  boundary components.

$\text{Map}_R$  the subgroup of the mapping class group  $\text{Map}_g$  generated by the Dehn twists  $t_r, r \in R$ .

$\text{Map}_R(-)$  the subsemigroup in  $\text{Map}_R$  generated by all conjugates of  $t_r, r \in R$  within  $\text{Map}_R$ .

$\text{Mon}$  the geometric monodromy group of a TLF, i.e. the image  $\text{Mon} := \text{mon}(\pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o))$  of the geometric monodromy representation.

$\text{mon}$  the geometric monodromy representation associated with a TLF of genus  $g$ , i.e.  $\text{mon} : \pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) \rightarrow \text{Map}_g$ .

$\mu$  the number of singular fibers in a TLF or a SLF.

$o$  a base point in  $S^2 \setminus \{q_1, \dots, q_\mu\}$ .

$\mathcal{O}_V$  the structure sheaf of an algebraic variety  $V$ .

$\mathcal{O}_V(1)$  a very ample line bundle on an algebraic variety  $V$ .

$f : X \rightarrow S^2$  a topological or symplectic Lefschetz fibration (TLF or SLF).

$f : X \rightarrow C_k$  a symplectic fibration of Lefschetz type over a higher genus surface.

$\pi_g^n$  the fundamental group of a smooth genus  $g$  surface with  $n$  punctures.

$\pi_n$  the fundamental group of a smooth compact surface of genus  $n$ .

$\pi_g \twoheadrightarrow \Gamma$  a finite presentation of  $\Gamma$ , i.e. a surjective homomorphism whose kernel is finitely generated as a normal subgroup.

$Q_i$  a critical point of a topological or symplectic Lefschetz fibration.

$q_i$  a critical value of a topological or symplectic Lefschetz fibration.

$R \subset C_g$  a graph connected collection of circles on the surface  $C_g$ .

$s \subset C_g$  a circle in  $C_g$  or in other words a smooth connected one dimensional submanifold in  $C_g$ .

$\Sigma$  a smooth projective algebraic curve.

$T_s$  a right handed Dehn twist diffeomorphism associated with a circle  $s \in C_g$ . In other words for any  $s \subset C_g$  one chooses an orientation preserving identification of a tubular neighborhood of  $s$  with the oriented cylinder  $[0, 1] \times S^1 \subset \mathbb{R} \times \mathbb{C}$  and then defines  $T_s \in \text{Diff}^+(C_g)$  as the diffeomorphism that acts a  $(t, z) \mapsto (t, e^{2\pi it} z)$  on the cylinder and as identity everywhere else.

$t_s$  the mapping class of a right handed Dehn twist  $T_s$ . The element  $t_s \in \text{Map}_g$  depends only on the isotopy class of  $T_s$ .

$U_i$  a small neighborhood of a critical value of a TLF or SLF.

$U_{Q_i}$  a small neighborhood of a critical point in a TLF or SLF.

$X_i$  the singular fiber of a TLF or a SLF corresponding to the critical value  $q_i$ .

$X_o$  a regular fiber of a TLF or a SLF.

## 2 Symplectic Lefschetz fibrations

First we recall some basic definitions and results. For more details the reader may wish to consult [SGA7II73, Section 3.2.7] and [Kas80].

**Definition 2.1** *Let  $X$  be a smooth compact 4-manifold equipped with a smooth surjective map  $f : X \rightarrow S^2$ . We shall call it a topological Lefschetz fibration (TLF) if the following conditions hold:*

- (i) *The differential  $df$  is surjective outside a finite subset of points  $\{Q_1, \dots, Q_\mu\} \subset X$ .*
- (ii) *Whenever  $p \in S^2 \setminus \{f(Q_1), \dots, f(Q_\mu)\}$  the fiber  $f^{-1}(p)$  is a smooth orientable Riemann surface of a given genus  $g$ .*
- (iii) *The images  $q_i := f(Q_i)$  are different for different  $Q_i$ .*
- (iv) *Let  $X_i$  denote the fiber of  $f$  containing  $Q_i$ . Then for any  $i$  there are small disks  $q_i \in U_i$  and  $Q_i \subset U_{Q_i} \subset X$  with  $f : U_{Q_i} \rightarrow U_i$  being a complex Morse function in some complex coordinates  $(x, y)$  on  $U_{Q_i}$  and  $z$  on  $U_i$ , i.e.  $z = f(x, y) = x^2 + y^2$ .*

A topological Lefschetz fibration  $f : X \rightarrow S^2$  will be called *orientable* (or *chiral*) if there exists an orientation on  $X$  so that the complex coordinates in (iv) above can be chosen in a way compatible with the orientations on  $X$  and  $S^2$ . Note that the definition of a topological Lefschetz fibration is designed in such a way that the function  $|f|^2$  is a Morse function near the singular fibers  $X_i$ . The Morse flow gives a handle body decomposition of  $X$  and in particular one gets standard retractions  $cr_i : f^{-1}(\overline{U_i}) \rightarrow X_i$ . Fix a base point  $o \in S^2$  which is a regular value of  $f$  and choose arcs  $a_1, \dots, a_\mu$  in  $S^2 \setminus (\cup_i U_i)$  which connect  $o$  with some point on  $\partial U_i$ ,  $i = 1, \dots, \mu$  as in Figure 2.1

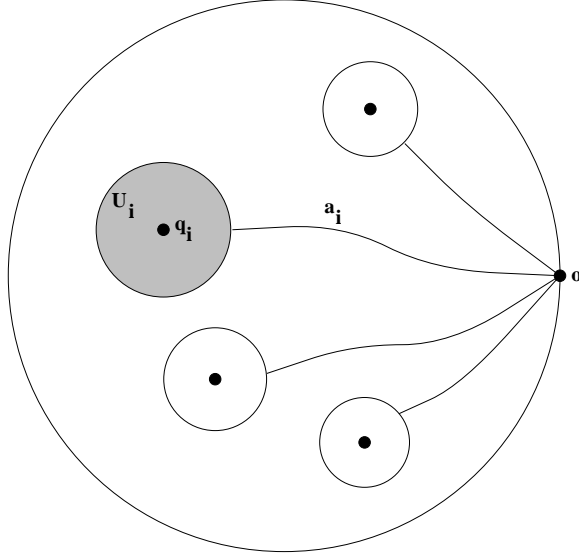


Figure 2.1: An arc system for  $f : X \rightarrow S^2$ .

Such a collection of arcs and discs is called an *arc system* for the TLF. A choice of an arc system gives a presentation of the fundamental group of  $S^2 \setminus \{q_1, \dots, q_\mu\}$ :

$$\pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) = \langle c_1, \dots, c_\mu \mid c_1 \cdot \dots \cdot c_\mu = 1 \rangle,$$

where geometrically  $c_i$  is represented by the  $o$ -based loop in  $S^2 \setminus \{q_1, \dots, q_\mu\}$  obtained by tracing  $a_i$  followed by tracing  $\partial U_i$  counterclockwise and then tracing back  $a_i$  in the opposite direction.

Since the family  $f : X \rightarrow S^2$  is locally trivial when restricted to each  $a_i$  we get well defined retractions of  $X_o$  onto each of the singular fibers  $X_i$ . By abuse of notation these retractions will be denoted by  $cr_i$  as well. Each  $cr_i : X_o \rightarrow X_i$  contracts a smooth circle  $s_i \subset X_o$  - the *geometric vanishing cycle*. The boundary of  $f^{-1}(U_i)$  is diffeomorphic to a smooth fiber bundle over the circle  $\partial U_i$  with  $X_o$  as a fiber. This fiber bundle is determined by a gluing diffeomorphism  $T_{s_i} : X_o \rightarrow X_o$  which is the usual Dehn twist along the circle  $s_i$ . More precisely  $T_{s_i}$  is the right handed Dehn twist along  $s_i$  with respect to the orientation on  $X_o$  compatible with the orientation on  $U_{Q_i}$  given by the complex coordinates around  $Q_i$ . The circles  $s_i$  and the Dehn twists  $T_{s_i}$  are uniquely determined up to a smooth isotopy and thus give well defined elements  $t_i \in \text{Diff}^+(X_o) / \text{Diff}_0^+(X_o) =: \text{Map}_g \subset \text{Out}(\pi_1(X_o))$  - the group of mapping classes of an oriented surface of genus  $g$ . The homomorphism  $\text{mon} : \pi_1(S^2 \setminus \{q_1, \dots, q_\mu\}, o) \rightarrow \text{Map}_g$ ,  $\text{mon}(c_i) = t_i$  is called the *geometric monodromy representation* of  $f : X \rightarrow S^2$  and its image is called the *geometric monodromy group* of the TLF. It is known [Kas80, Theorem 2.4] that if  $f : X \rightarrow S^2$  is an orientable TLF of genus  $g \geq 2$ , then the geometric monodromy representation of  $f$  uniquely determines the diffeomorphism type of  $f$ .

Note that if  $C_g$  is an oriented surface and  $s \subset C_g$  is a smoothly embedded, homotopically non-trivial circle one can perform both the right handed Dehn twist  $T_s : C_g \rightarrow C_g$  and the left handed Dehn twist  $T_s^{-1} : C_g \rightarrow C_g$ . If however  $f : X \rightarrow S^2$  is an orientable TLF and  $C_g = X_o$  is given the

induced orientation from  $X$ , then all of the geometric monodromy transformations  $\{T_{s_1}, \dots, T_{s_\mu}\}$  are right handed Dehn twists. This property actually characterizes the orientable TLF completely.

We also consider topological Lefschetz fibrations which are compatible with an additional closed non-degenerate 2-form  $w$  on  $X$ .

**Definition 2.2** *Assume that  $f : X \rightarrow S^2$  is a TLF and that  $(X, w)$  is a symplectic manifold. We say that  $(f : X \rightarrow S^2; w)$  is a symplectic Lefschetz fibration (SLF) if for any  $p \in S^2$  the form  $w$  is non-degenerate on the fiber  $X_p$  at  $p$  in the sense that the smooth locus of  $X_p$  is a symplectic submanifold in  $X$  and for every  $i$  the symplectic form  $w_{Q_i}$  is non-degenerate on each of the two planes contained in the tangent cone of  $X_i$  at  $Q_i$ .*

Gompf [Gompf-Stipsicz] had shown that under some mild restrictions the SLF can also be characterized in purely topological terms. For the convenience of the reader we recall the proof of this very useful fact. Different proofs can be found in [Gompf-Stipsicz] or [Smith98].

**Proposition 2.3 (R.Gompf)** *A topological Lefschetz fibration  $f : X \rightarrow S^2$  of curves of genus  $g \geq 2$  admits a symplectic structure if and only if it is orientable.*

**Proof.** Let us recall first the necessity of the orientation restriction. Assume that a symplectic Lefschetz fibration  $f : X \rightarrow S^2$  has two singular values  $q_1, q_2$  with singular points  $Q_1, Q_2$  above them, such that the monodromy Dehn twists around  $q_1, q_2$  have opposite orientations, i.e. there exists a complex chart around  $Q_1 \in X$  such that  $Q_1 = (0, 0)$ ,  $f(x_1, y_1) = x_1^2 + y_1^2$  and another chart centered at  $Q_2$  such that  $f(x_2, y_2) = \bar{x}_2^2 + \bar{y}_2^2$ . Here both charts are chosen to be compatible with the orientation on  $X$  given by the symplectic structure  $w$ . By the definition of a SLF there is a symplectic form  $w_{\text{base}}$  on  $S^2$  induced from  $w$  on  $X$ .

Join  $q_1, q_2$  by a segment  $I$  in the base  $S^2$ . The fibration  $f$  restricts to a trivial family over the interior of  $I$  with vanishing loop contractions at the endpoints. Take a lift of the segment  $I$  to  $X$  with beginning and end points in the above coordinate charts, and choose a parallel trivial family of horizontal tangent planes  $\{\pi_h(s)\}$  over this lifted segment. If we pick a trivial frame  $e_1(s), e_2(s)$  for this family of planes we see that  $w_{\text{base}}(e_1(s), e_2(s))$  changes sign going from  $q_1$  to  $q_2$ . This contradicts the continuity of the frame  $e_1(s), e_2(s)$  and so the Dehn twists around  $q_1, q_2$  must have the same orientation.

We proceed to show that this condition is sufficient.

Let  $f : X \rightarrow S^2$  be an orientable TLF with singular values  $q_1, \dots, q_\mu$ . Take a topologically simple cover of  $S^2$  by open disks  $\{D_\alpha\}$ , such that it includes a disk  $U_i$  centered at every singular value  $q_i$ , and  $q_i \notin \bar{U}_\alpha$  if  $\alpha \neq i$ . We will put symplectic structures on the families over the disks  $U_\alpha$  first, and then glue them adapting an argument of Thurston (see [McDuff-Salamon98, Theorem 6.3]) for symplectic fibrations.

For every disk  $D_i$  containing a singular value, take the trivial family  $C_g \times U_i \rightarrow U_i$ , endowed with a symplectic form  $w_{\text{fiber}} \oplus w_{\text{base}}$ , the summands being symplectic forms on the factors. By identifying  $C_g$  with a regular fiber of  $f|_{f^{-1}D_i}$  and choosing an arc from the basepoint regular value to the singular value of the fibration we get a vanishing loop  $s_i \subset C_g$  that determine the diffeomorphism type of the pencil  $f$  over  $D_i$ . We will perform symplectic surgery on the trivial family  $C_g \times U_i$  to make it diffeomorphic to  $f$ .

Let  $q : \mathbb{C}^2 \rightarrow \mathbb{C}$  be the standard quadratic map  $q(x, y) = x^2 + y^2$ . Take a small ball  $B$  centered at  $(0, 0) \in \mathbb{C}^2$ , a disk  $D \subset q(B)$  such that the 3-sphere  $\partial B$  is transverse to the fibers of  $q$  and such

that  $B = q^{-1}(D) \cap \overline{B}$ . The restricted family  $q|_B : B \rightarrow D$  has a single quadratic singular fiber  $B_0$  over  $0 \in D \subset \mathbb{C}$ .

Let  $s \subset C_g$  be one of the vanishing simple loops of the fibration  $X$ . Consider the normalization  $\tilde{B}_0 \rightarrow B_0$  and let  $o_1$  and  $o_2$  be the preimages of the singular point  $(0, 0) \in B_0 \subset B$  in  $\tilde{B}_0$ . Choose small analytic discs in  $\tilde{B}_0$  centered at  $o_1$  and  $o_2$  respectively and let  $\Delta_1, \Delta_2$  denote their images in  $B_0$ .

Select two open annuli  $A_1 \subset \Delta_1 \setminus \{(0, 0)\}$  and  $A_2 \subset \Delta_2 \setminus \{(0, 0)\}$  such that the point  $(0, 0)$  does not lie in the closure  $\bar{A}_1 \cup \bar{A}_2$  (see Figure 2.2). Using Moser's characterization of symplectic type of surfaces by volume, we may choose also two open cylinders  $C_1, C_2$  on the opposite sides of a bicollar neighborhood of  $c$ , such that both  $C_1, C_2$  are retracts of the bicollar neighborhood, their adherence  $\bar{C}_1 \cup \bar{C}_2$  does not intersect  $c$ , and they are symplectomorphic to  $A_1, A_2$  respectively.

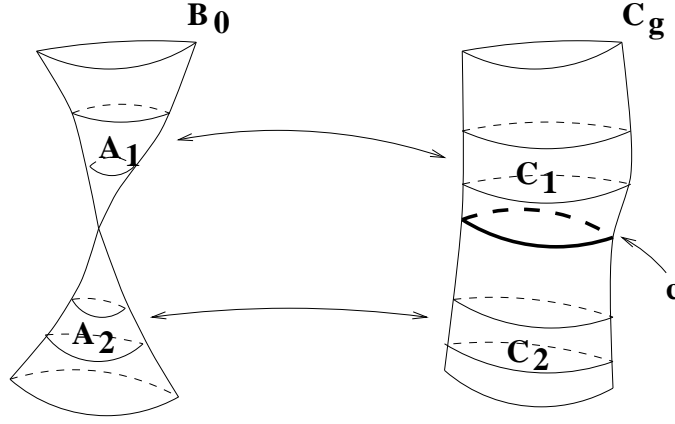


Figure 2.2: The cylinders  $A_i$  and  $C_i$ ,  $i = 1, 2$ .

The annuli  $A_1, A_2, C_1, C_2$  are embedded in their respective total spaces with trivial normal bundle, so by Weinstein's symplectic neighborhood theorem [Gompf95, Lemma 2.1], there exists an  $\varepsilon > 0$  and open neighborhoods  $W_i$  of  $A_i$  in  $B$ ,  $V_j$  of  $C_j$  in  $C_g \times S^2$  for  $i, j = 1, 2$  such that the  $W_i, V_j$  are symplectomorphic to  $A_i \times D_\varepsilon, C_j \times D_\varepsilon$  respectively. Shrinking  $D$  if necessary, we may now perform a surgery by inserting the Dehn twist of  $q|_B : B \rightarrow D$  by the identifications  $W_i \cong V_i$ . The symplectic structures on  $B$  and  $C_g \times D_i$  define a symplectic structure on  $f$  over  $D_i$  in this way.

We repeat this surgery over all the critical values of  $f$ , rescaling the obtained symplectic structures  $\omega_1, \dots, \omega_\mu$  so that they induce the same symplectic structure  $[\sigma]$  on the regular fibers of the pencil. Through diffeomorphisms with trivial families we may also endow the restrictions of  $f$  over the regular disks  $U_\alpha$  with symplectic structures  $\omega_\alpha$  inducing the same symplectic structure  $[\sigma]$  on the regular fiber of the pencil as the twist families over the singular values.

Let now  $\tau_0 \in \Omega^2(X)$  be a closed 2-form such that its restriction to the fibers represents the cohomology class  $[\sigma]$ . Over every disk  $D_\alpha$  we have that  $\tau_0$  and the symplectic form  $\omega_\alpha$  are cohomologous, thus we may select 1-forms  $\lambda_\alpha \in \Omega^1(f^{-1}(D_\alpha))$  so that

$$\omega_\alpha - \tau_0 = d\lambda_\alpha$$

Choose now a partition of unity  $\{\rho_\alpha : D_\alpha \rightarrow [0, 1]\}$  subordinate to the cover  $\{U_\alpha\}$  and such that for every critical value  $q_i$  the function  $\rho_i$  has constant value 1 on its neighbourhood. Define  $\tau \in \Omega^2(X)$



by

$$\tau = \tau_0 + \sum_{\alpha} d((\rho_{\alpha} \circ f)\lambda_{\alpha}).$$

This is a closed form, cohomologous to  $\tau_0$ , restricting to the class  $[\sigma]$  in every fiber and equal to the previously found symplectic  $\omega_i$  in neighbourhoods of the singular fibers. As in the case of smooth symplectic fibrations, this form  $\tau$  is nondegenerate on the tangent spaces to the fibers, and as it is defined on the compact total space, if we select a symplectic form  $\beta \in \Omega^2(S^2)$  the forms

$$\omega_K = \tau + Kf^*\beta$$

are nondegenerate for sufficiently large  $K$ .

□

### 3 The main construction

#### 3.1 Positive relations among right Dehn twists

Let  $g$  be a nonnegative integer. Fix a compact oriented reference surface  $C_g$  of genus  $g$  and an infinite sequence of distinct points  $x_0, x_1, \dots, x_n, \dots \in C_g$ . Put  $\pi_g^n := \pi_1(C_g \setminus \{x_1, \dots, x_n\}, x_0)$  and let  $\text{Diff}^+(C_g)_r^n$  denote the group of all orientation preserving diffeomorphisms of  $C_g$  that fix the points  $x_1, x_2, \dots, x_{n+r}$  and induce the identity on the tangent spaces  $T_{x_i}C_g$  for  $i = n+1, \dots, n+r$ . For any triple of non-negative integers  $(g, n, r)$  such that  $2g - 2 + n + 2r > 0$  define the *mapping class group*  $\text{Map}_{g,r}^n$  as the group of connected components of  $\text{Diff}^+(C_g)_r^n$ , i.e.

$$\text{Map}_{g,r}^n := \pi_0(\text{Diff}^+(C_g)_r^n).$$

As usual we will skip the labels  $n$  and  $r$  if they happen to be equal to zero.

By definition the mapping class group  $\text{Map}_g^n$  acts by outer automorphisms on  $\pi_g^n$ . In fact this action identifies [Hain-Looijenga97]  $\text{Map}_g$  with the index two subgroup of  $\text{Out}(\pi_g)$  consisting of outer automorphisms acting trivially on  $H^2(\pi_g, \mathbb{Z}) \cong H^2(C_g, \mathbb{Z})$ . Similarly one can interpret the group  $\text{Map}_g^1$  as the group of all automorphisms of  $\pi_g$  acting trivially on  $H^2(\pi_g, \mathbb{Z})$ . The group  $\text{Map}_{g,r}^n$  is generated by the right handed Dehn twists along all (unoriented) non separating loops  $c \subset C_g \setminus \{x_1, \dots, x_{n+r}\}$ .

**Remark 3.1** For any  $g, n$  the natural forgetful map  $\text{Map}_{g,n} \rightarrow \text{Map}_g^n$  is surjective and has a central kernel which can be identified with the free abelian group generated by the Dehn twists along simple loops around the punctures.

For future reference define  $\text{Map}_{g,r}^n(-) \subset \text{Map}_{g,r}^n$  to be the sub-semigroup of  $\text{Map}_{g,r}^n$  generated by (the images of) all right handed Dehn twists.

The first step in the construction is the following simple observation.

**Lemma 3.2** *Let  $s_1, \dots, s_m \subset C_g$  be free simple closed loops (not necessarily distinct) and let  $t_1, \dots, t_m \in \text{Map}_g$  be the corresponding right handed Dehn twists. Suppose that there exist integers  $\{n_i\}_{i=1}^m$  so that the  $t_i$ 's satisfy the relation  $\prod_i t_i^{n_i} = 1$  in the mapping class group. Then there exists a TLF  $f : X \rightarrow S^2$  with the following properties:*

- (a) *The regular fiber of  $f$  can be identified with  $C_g$  so that the vanishing cycles of  $f$  are precisely the  $s_i$ 's.*
- (b) *The geometric monodromy group  $\text{Mon}(f)$  of  $f$  is just the subgroup of the mapping class group generated by the  $t_i$ 's.*
- (c) *The fundamental group of  $X$  is isomorphic to the quotient of  $\pi_g$  by the normal subgroup generated by all the  $s_i$ 's.*
- (d) *If all the  $n_i$ 's are positive, then  $f$  is a SLF.*

**Proof.** To construct the fibration  $f : X \rightarrow S^2$  start with the direct product  $D_0 \times C_g$  where  $D_0 \subset \mathbb{C}$  is a small disk around zero. Next attach  $|n_i|$ -copies of small discs  $D_i$ ,  $i = 1, \dots, n$ , along the boundary of  $D_0$ . Over each  $D_i$  choose a standard holomorphic Lefschetz fibration  $U_{X_i} \rightarrow D_i$  with a unique singular fiber  $X_i$  at the center so that the vanishing cycle is exactly  $s_i$ . The union of  $D_0 \times C_g$  and the fibrations  $U(X_i)$  (each appearing  $|n_i|$ -times respectively) has a structure of a topological Lefschetz fibration  $u : U \rightarrow D$  over a larger disc  $D$ . By construction the Lefschetz fibration  $u$  has a regular fiber isomorphic to  $C_g$  and vanishing cycles  $s_1, \dots, s_m$ . Moreover the geometric monodromy transformation  $\text{mon}(\partial D) \in \text{Map}_g$  for  $u$  is precisely the product  $\prod_i t_i^{n_i}$ . Since the latter is equal to identity in  $\text{Map}_g$  we get that  $U|_{\partial D}$  is homotopy equivalent (and hence diffeomorphic) to a product  $C_g \times S^1$ . In particular we can extend  $u$  to a TLF over  $S^2$ .

The conditions (a) and (b) are satisfied by construction. Let  $f^\# : X^\# \rightarrow S^\#$  denote the fibration obtained from  $f$  by removing the singular fibers. Since  $f^\#$  is a fiber bundle and by construction  $f$  admits a topological section we can identify the fundamental group  $\pi_1(X^\#)$  with the semidirect product  $\pi_1(S^\#) \ltimes_{\text{mon}} \pi_g$  where  $\text{mon} : \pi_1(S^\#) \rightarrow \text{Aut}(\pi_g)$  is the monodromy representation. Now the Seifert-van Kampen theorem implies that  $\pi_1(X)$  is isomorphic to the quotient of  $\pi_g$  by the normal subgroup generated by the orbits of the vanishing loops  $\{s_1, \dots, s_m\}$  under the monodromy group  $\text{Mon}$  of  $f$ . On the other hand any Dehn twist  $t_s$  is uniquely characterized by the property that the maximal quotient of  $\pi_g$  on which  $t_s$  acts as the identity is the quotient of  $\pi_g$  by the normal subgroup generated by  $s$ . In particular for any  $\gamma \in \pi_g$  the element  $\gamma^{-1}t_s(\gamma)$  is a product of conjugates of  $s$  and so the normal subgroup of  $\pi_g$  generated by the  $\text{Mon}$ -orbits of  $s_1, \dots, s_m$  coincides with the normal subgroup generated by  $s_1, \dots, s_m$  only which proves part (c) of the Lemma.

Finally the fact that condition (d) holds for  $f$  is a consequence of Proposition 2.3. The Lemma is proven.  $\square$

The previous lemma shows that the construction of symplectic Lefschetz fibrations reduces to the problem of finding relations in the semigroup  $\text{Map}_g(-) \subset \text{Map}_g$ . In order to find such relations and to be able to modify them we will need to study certain configurations of embedded circles in  $C_g$ .

Recall [Ivanov-McCarthy95] that given two isotopy classes  $\rho$  and  $\sigma$  of smooth circles in  $C_g$  one defines the *geometric intersection number*  $i(\rho, \sigma)$  as the minimum number of points of  $r \cap s$  over all representatives  $r$  of  $\rho$  and  $s$  of  $\sigma$ . A finite set  $R$  of smooth circles in  $C_g$  is said to be in *minimal position* if every two elements  $r, s \in R$  intersect transversally in exactly  $i([r], [s])$  points and no three elements in  $R$  intersect.

Let  $G_R$  be the dual graph of the one dimensional cell complex  $\cup_{r \in R} r \subset C_g$ , i.e. the graph whose vertices are the elements of  $R$  and for which the edges connecting two vertices  $s, r \in R$  correspond to the intersection points of the loops  $s$  and  $r$  in  $C_g$ .

**Definition 3.3** (a) Two loops  $s, r \in R$  will be called adjacent if they intersect transversally at a single point, i.e. if the vertices  $s$  and  $t$  in  $G_R$  are connected by a single edge.

(b) Two circles  $s, r \in R$  will be called graph connected if the corresponding vertices are connected by a path of edges in  $G_R$ . In other words  $s, r \in R$  are graph connected if there exist a sequence of circles  $s_1, \dots, s_m \in R$  so that  $s = s_1$ ,  $r = s_m$  and  $s_i$  is adjacent to  $s_{i+1}$  for all  $i = 1, \dots, m - 1$ .

(c) Let  $R$  and  $S$  be two finite sets of circles in  $C_g$  so that  $R \cup S$  is in minimal position. We will say that  $R$  is graph connected to  $S$  if for any loop  $r \in R$  there exists a loop  $s \in S$  so that  $r$  and  $s$  are graph connected in  $R \cup S$ .

**Remark 3.4** The adjacency condition in part (b) of Definition 3.3 is imposed only on two consecutive loops in the sequence  $s_1, \dots, s_m$ . In particular, non-consecutive loops may intersect in an arbitrary transverse fashion.

For any pair of smooth circles  $a, b \subset C_g$  intersecting at exactly one point one can find a neighborhood  $C_{a,b}$  of  $a \cup b$  in  $C_g$  diffeomorphic to a torus with one hole as depicted on Figure 3.1.

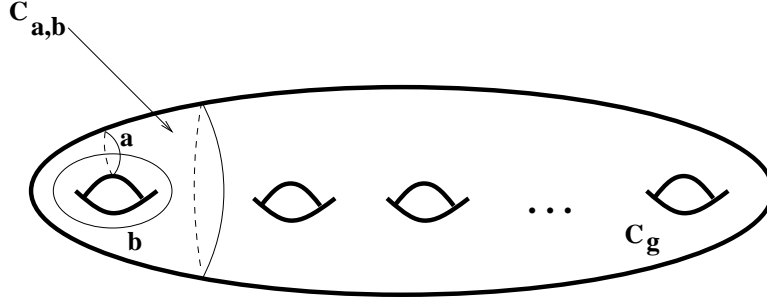


Figure 3.1: A handle determined by two adjacent cycles.

Thus we obtain a homomorphism  $h_{a,b} : \text{Map}_{1,1} \rightarrow \text{Map}_g$  defined by the handle  $C_{a,b} \subset C_g$ .

For a set  $R$  of circles in minimal position denote by  $\text{Map}_R$  the subgroup of  $\text{Map}_g$  generated by the right Dehn twists  $\{t_r\}_{r \in R}$ . Let  $\text{Map}_R(-)$  denote the subsemigroup in  $\text{Map}_R$  generated by the right twists  $\{t_r\}_{r \in R}$  and all of their conjugates in  $\text{Map}_R$ . Note that  $\text{Map}_R(-)$  is contained in  $\text{Map}_g(-)$  as a subsemigroup since for every  $\phi \in \text{Map}_g$  and every isotopy class  $\rho$  of circles on  $C_g$  one has  $\phi \circ t_\rho \circ \phi^{-1} = t_{\phi(\rho)}$ . Moreover by definition the semigroup  $\text{Map}_R(-) \subset \text{Map}_R$  is invariant under conjugation in  $\text{Map}_R$ .

The next two lemmas give a topological characterization of the existence of relations in  $\text{Map}_g(-)$ .

**Lemma 3.5** Let  $L$  be a finite set of circles in minimal position in  $C_g$ . Then  $\text{Map}_L(-) = \text{Map}_L$  if and only if there exists a finite relation  $\alpha = 1$  where  $\alpha$  is a product of only positive powers of  $t_{\phi(l)}$ 's with  $l \in L$ ,  $\phi \in \text{Map}_L$  and each  $t_l, l \in L$  occurs at least once in  $\alpha$ .

**Proof.** For the proof of the “if” part consider a relation  $\alpha = 1$  as in the statement of the lemma. By hypothesis there exist a positive integer  $\mu$  and a surjective map  $l : \{1, 2, \dots, \mu\} \rightarrow L$  so that

$$1 = \alpha = \prod_{i=1}^{\mu} t_{l(i)}$$

in  $\text{Map}_L$ . If we multiply both sides of the above relation by  $t_{l(1)}^{-1}$  on the left we get that

$$t_{l(1)}^{-1} = \prod_{i=2}^{\mu} t_{l(i)} \in \text{Map}_L(-).$$

Since the relation is cyclic this implies that  $t_{l(i)}^{-1} \in \text{Map}_L(-)$  for all  $i = 1, \dots, \mu$ . Combining this with the surjectivity of  $l$  yields the inclusion  $\{t_l\}_{l \in L} \subset \text{Map}_L(-)$  and hence  $\text{Map}_L(-) = \text{Map}_L$ .

To prove the “only if” part of the lemma note that the assumption  $\text{Map}_L(-) = \text{Map}_L$  implies that  $t_l^{-1} \in \text{Map}_L(-)$  for all  $l \in L$ . In other words each  $t_l^{-1}$  can be written as a product of finitely many of the right Dehn twists  $\{t_{\phi(s)}\}_{s \in L, \phi \in \text{Map}_L}$ . Let now  $m$  be the cardinality of  $L$  and let  $l_1, \dots, l_m$  be an ordering of  $L$ . Next take the obvious relation

$$1 = t_{l_1} t_{l_1}^{-1} t_{l_2} t_{l_2}^{-1} \dots t_{l_m} t_{l_m}^{-1}$$

and then replace each of the left Dehn twists  $t_{l_i}^{-1} \in \text{Map}_L = \text{Map}(-)$  with the corresponding product of right Dehn twists. The resulting right hand side will be a word  $\alpha$  with the desired property.  $\square$

The next lemma gives a convenient criterion guaranteeing the existence of positive relations among right Dehn twists.

**Lemma 3.6** *Let  $R$  and  $S$  be two sets of circles in minimal position which are graph connected. Assume that  $\text{Map}_S(-) = \text{Map}_S$ . Then  $\text{Map}_{R \cup S}(-) = \text{Map}_{R \cup S}$ .*

**Proof.** By definition  $\text{Map}_{R \cup S}(-)$  is the subsemigroup in  $\text{Map}_g$  generated by  $t_{\phi(c)}$ ,  $\phi \in \text{Map}_{R \cup S}$ ,  $c \in R \cup S$ . Therefore we need to show that for any  $\phi \in \text{Map}_{R \cup S}$  and  $c \in R$  the left twist  $t_{\phi(c)}^{-1}$  also belongs to  $\text{Map}_{R \cup S}(-)$ . Since  $t_{\phi(c)} = \phi \circ t_c \circ \phi^{-1}$  and  $\text{Map}_{R \cup S}(-)$  is conjugation invariant in  $\text{Map}_{R \cup S}$  it suffices to check that  $t_c^{-1} \in \text{Map}_{R \cup S}(-)$  whenever  $c \in R \cup S$ .

If  $c \in S$  then by assumption  $t_c^{-1} \in \text{Map}_S(-) \subset \text{Map}_{R \cup S}(-)$ . If  $c \in R$ , then by the hypothesis of the lemma  $c$  is graph connected in  $G_{R \cup S}$  with some element  $d \in S$ . On the other hand according to Lemma 5.1 (b) for any pair  $a, b$  of adjacent cycles on  $C_g$  the right twists  $t_a, t_b$  are conjugate in  $h_{a,b}(\text{Map}_{1,1}) \subset \text{Map}_{R \cup S}$ . In particular  $t_c^{-1}$  and  $t_d^{-1}$  belong to the same conjugacy class in  $\text{Map}_{R \cup S}$ . Since by assumption  $t_d^{-1} \in \text{Map}_S(-) \subset \text{Map}_{R \cup S}(-)$  and due to the conjugation invariance of  $\text{Map}_{R \cup S}(-)$  in  $\text{Map}_{R \cup S}$  we conclude that  $t_c^{-1} \in \text{Map}_{R \cup S}(-)$  as well, which concludes the proof of the lemma.  $\square$

Let now  $\Gamma$  be a finitely presentable group and let  $a : \pi_g \twoheadrightarrow \Gamma$  be a given presentation. As we can see from the previous lemmas the construction of a SLF whose total space has fundamental group  $\Gamma$  reduces to finding a graph connected system of circles  $R$  so that  $R$  generates  $\ker(a)$  as a normal subgroup and the right Dehn twists about some subsystem of  $S$  satisfy a positive relation in  $\text{Map}_g$ .

To achieve this we will have to modify the presentation  $a$  and in particular enlarge the genus  $g$ . This will be done in two steps which are explained in the next two sections.

### 3.2 Geometric presentations

The first step is purely topological. Starting with a presentation  $a : \pi_g \rightarrow \Gamma$  we show how to add handles to  $C_g$  to obtain a new presentation  $\psi : \pi_e \rightarrow \Gamma$  for which the cycles generating  $\ker(\psi)$  are nicely situated on  $C_e$ .

We begin with the following definition:

**Definition 3.7** *Let  $\Gamma$  be a finitely presentable group. A geometric presentation of  $\Gamma$  is a surjective homomorphism  $\psi : \pi_e \rightarrow \Gamma$ , where  $\pi_e$  is the fundamental group of a compact Riemann surface  $C_e$  of genus  $e$  such that:*

- (a) *The group of relations  $\ker(a)$  is generated as a normal subgroup by finitely many simple closed loops  $r_1, \dots, r_m \subset C_e$  ;*
- (b) *Every two of the loops  $r_1, \dots, r_m$  intersect transversally at most at one point and the subspace  $\cup_i r_i \subset C_e$  is connected.*

As we will see in the next section the geometrically presented groups are well suited for algebraic geometric manipulations. Thus it is important to find a procedure for constructing geometric presentations of finitely presentable groups. We have the following

**Lemma 3.8** *Let  $a : \pi_g \rightarrow \Gamma$  be a given presentation. Then there exists a surjective homomorphism  $\pi_e \twoheadrightarrow \pi_g$  so that the composition  $\pi_e \rightarrow \pi_g \rightarrow \Gamma$  is a geometric presentation.*

**Proof.** Let  $R \subset \ker(a)$  be a finite subset of elements which generates  $\ker(a)$  as a normal subgroup (such a set exists since by assumption  $a$  is a finite presentation of  $\Gamma$ ). Represent each element  $r \in R$  by a free immersed loop  $c_r \subset C_g$  so that the one dimensional simplicial sub complex  $M := \cup_{r \in R} c_r \subset C_g$  is connected and has only ordinary double points as singularities. Such a collection of immersed loops  $c_r$  can be easily found as a small perturbation of a standard representation of the elements in  $R$  as immersed loops based at some fixed point  $x_0 \in C_g$ .

For each singular point  $p \in M$  choose a small disk  $p \in D_p \subset C_g$  which doesn't contain any other singularity of  $M$ . Now for each  $p \in \text{Sing}(M)$  delete from  $D_p$  a smaller disk centered at  $p$  and glue a handle  $C_p$  to  $C_g$  along the inner rim of the resulting annulus. The two branches of  $M \cap D_p$  meeting at  $p$  can be then completed to two smooth disjoint curves on  $C_p$  as shown on Figure 3.2.

In this way we obtain a new smooth surface

$$C_e := (C_g \setminus \text{Sing}(M)) \cup (\cup_{p \in \text{Sing}(M)} C_p)$$

of genus  $e = g + \#(\text{Sing}(M))$  and a system of smooth disjoint circles  $\tilde{c}_r \subset C_e$ ,  $r \in R$ . Next for every  $p \in \text{Sing}(M)$  we choose standard generators  $a_p, b_p \subset C_p \subset C_e$  of the fundamental group of  $C_p$  as on Figure 3.2. By construction  $\Gamma$  is isomorphic to the quotient of  $\pi_e$  by the normal subgroup  $N \triangleleft \pi_e$  generated by the finite set of elements

$$\{\tilde{c}_r\}_{r \in R} \cup (\cup_{p \in \text{Sing}(M)} \{a_p, b_p\}) \subset \pi_e.$$

But we have glued the handles in such a way that the  $\tilde{c}_r$ 's are disjoint from each other,  $(a_p \cup b_p) \cap (a_q \cup b_q) \neq \emptyset$  only if  $p = q$  and each  $a_p$  or  $b_p$  intersects exactly one of the  $\tilde{c}_r$ 's transversally at a single point. Finally since every  $a_p$  intersects  $b_p$  at one point we conclude that the union of all these cycles is connected in  $C_e$  and hence  $\pi_e \rightarrow \pi_e/N \cong \Gamma$  is a geometric presentation.  $\square$

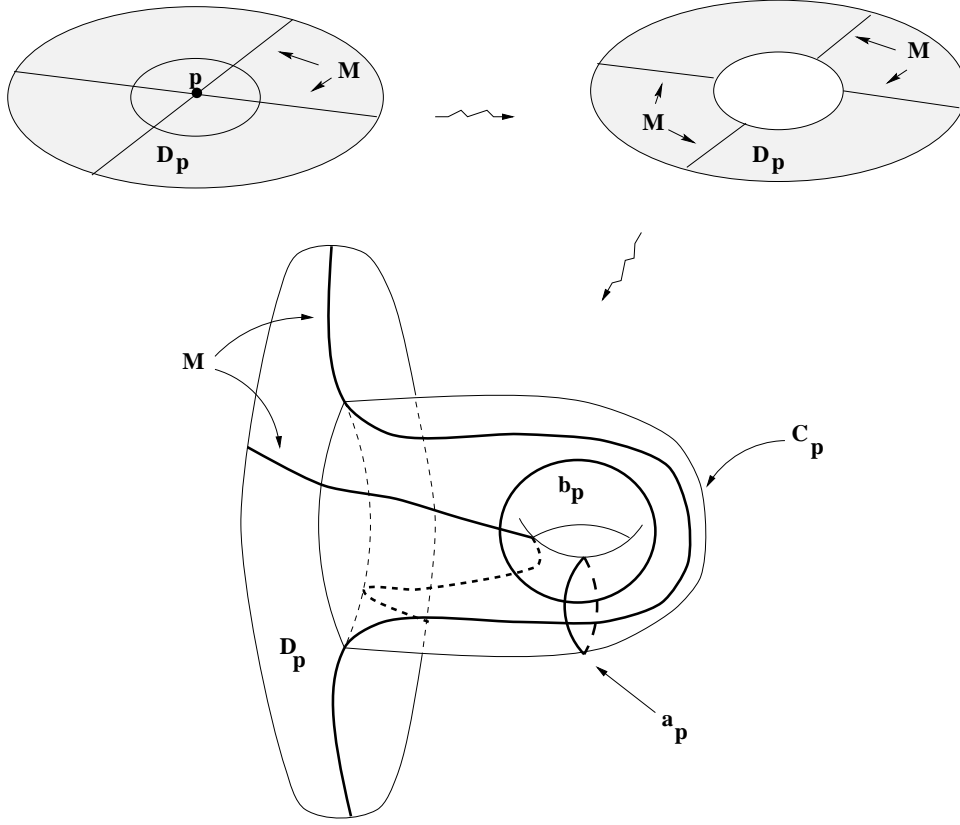


Figure 3.2: Gluing the handle  $C_p$  to  $D_p \setminus \{p\}$ .

### 3.3 The proof of Theorem A

Assume now that  $\Gamma$  is a group with a fixed geometric presentation  $\psi : \pi_e \rightarrow \Gamma$ . Let  $R = \{r_1, \dots, r_m\}$  be a set of circles in minimal position in  $C_e$  so that  $R$  generates  $\ker(a)$  as a normal subgroup as in Definition 3.7. Note that without a loss of generality we may assume that  $R$  contains a non-separating circle  $s \in R$ . Indeed, if  $\ker(a)$  happens to be contained in  $[\pi_e, \pi_e]$  we can always glue an extra handle to  $C_e$  and add to  $R$  the three standard generators of the first homology of the handle (plus some extra loops if required).

Let  $\Sigma$  be a smooth projective algebraic curve of genus  $e$  which we have identified as a  $C^\infty$  manifold with  $C_e$ . Fix a point  $p \in s \subset \Sigma$  which does not lie on any of the circles in  $R \setminus \{s\}$  and let  $p \in \Delta \subset \Sigma$  be a small analytic disc which is disjoint from all the circles in  $R \setminus \{s\}$ .

Let  $V = \Sigma \times \mathbb{P}^1$  and let  $D \subset V$  be a very ample divisor such that  $D = \sum_i D_i$  with each  $D_i \subset V$  being a section for the projection  $p_\Sigma : V \rightarrow \Sigma$ . By replacing  $\mathcal{O}_V(D)$  by its third power if necessary we may further assume that the degree of  $D$  on  $\mathbb{P}^1$  is divisible by three.

Let  $p_i$ ,  $\Delta_i$  and  $s_i$  denote the preimages in  $D_i$  of  $p$ ,  $\Delta$  and  $s$  respectively. Similarly let  $R_i$  be the set of circles in  $D_i$  consisting of the preimages of the circles in  $R$  via the projection  $p_{\Sigma|D_i} : D_i \rightarrow \Sigma$ . When  $\mathcal{O}_V(D)$  is chosen to be sufficiently ample and the divisor  $D$  is chosen to be a generic deformation of a set of sections of  $p_\Sigma$  which pass through a given point lying over  $p \in \Sigma$  we may

easily arrange that all intersections of the  $D_i$ 's are transverse and that for every pair of indices  $i \neq j$  we have  $\Delta_i \cap \Delta_j \neq \emptyset$  (see Figure 3.3 below).

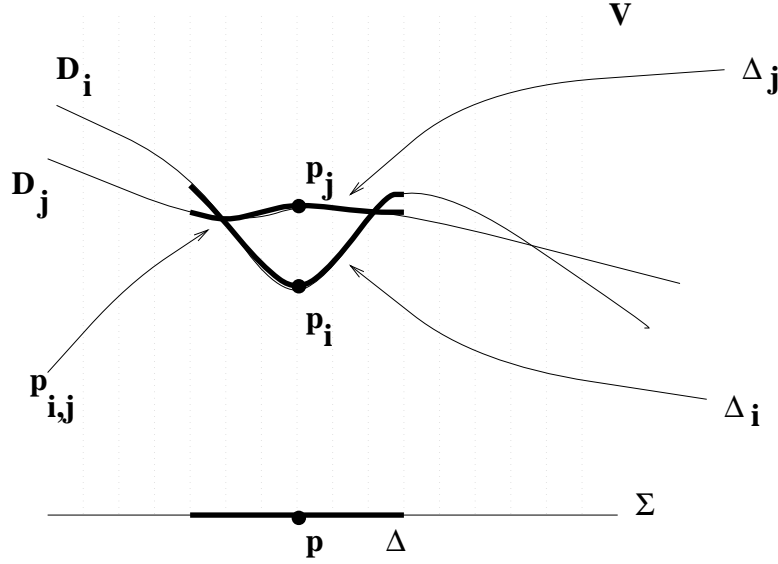


Figure 3.3: The divisors  $D_i$ .

For any  $i \neq j$  pick a point  $p_{i,j} \in \Delta_i \cap \Delta_j$ . Choose arcs  $a_{i,j}^i \subset \Delta_i$  connecting  $p_i$  with  $p_{i,j}$  which meet only at  $p_i$  and do not intersect  $s_i$  at any other point (see Figure 3.4).

**Remark 3.9** For each  $i$  and  $j$  only *one* point is chosen in  $\Delta_i \cap \Delta_j$ . Therefore the set of points  $\{p_{i,j}\}$  will be only a subset in  $\cup_{i < j} (D_i \cap D_j)$  in general.

Consider a generic pencil in the linear system  $|D|$  which contains  $D$  as a member and has a smooth general member. After blowing up the base points of this pencil on  $V$  we obtain a smooth surface  $\widehat{V}$  and a projective morphism  $v : \widehat{V} \rightarrow \mathbb{CP}^1$  having  $D$  as a fiber over some point  $d \in \mathbb{CP}^1$  and a smooth connected general fiber. Let  $V_o$  be a smooth fiber of  $v$  over some reference point  $o \in \mathbb{CP}^1$  which is close to  $d$ . Then we have (once we choose an arc system for  $v$ ) a well defined deformation retraction  $cr : V_o \rightarrow D$  which collapses certain smooth circles on  $V_o$  to the points in the finite set  $\cup_{i < j} (D_i \cap D_j)$ . Note that trough the retraction  $cr$  we may view the  $\Delta_i \setminus \{p_{i,j}\}_{j \neq i}$  as punctured discs on the smooth curve  $V_o$ , the set  $\cup_i R_i$  as a set of circles in minimal position on  $V_o$  and the arcs  $r_{i,j}^i$  as arcs on  $V_o$ . Moreover by a slight perturbation within each  $\Delta_i$  we can arrange that for each pair of indices  $i \neq j$  the arcs  $r_{i,j}^i, r_{i,j}^j \in V_o$  in joins smoothly at a point on the circle  $c_{i,j}$ .

In particular we see that the points  $p_{i,j} \in D$  correspond to vanishing cycles  $c_{i,j} \subset V_o$  for the pencil  $v$  which are disjoint from  $s_i \subset V_o$ . Moreover by a slight perturbation within each  $\Delta_i$  we can arrange that for each pair of indices  $i \neq j$  the arcs  $r_{i,j}^i, r_{i,j}^j \in V_o$  in joins smoothly at a point on the circle  $c_{i,j}$  (see Figure 3.5).

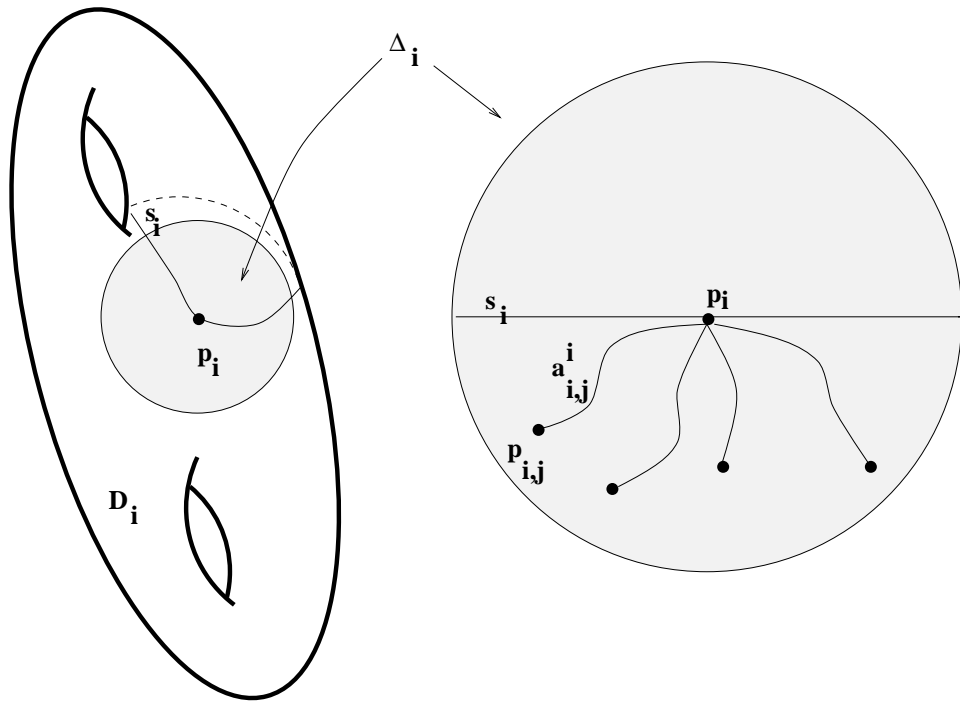


Figure 3.4: The system of arcs in  $\Delta_i$ .

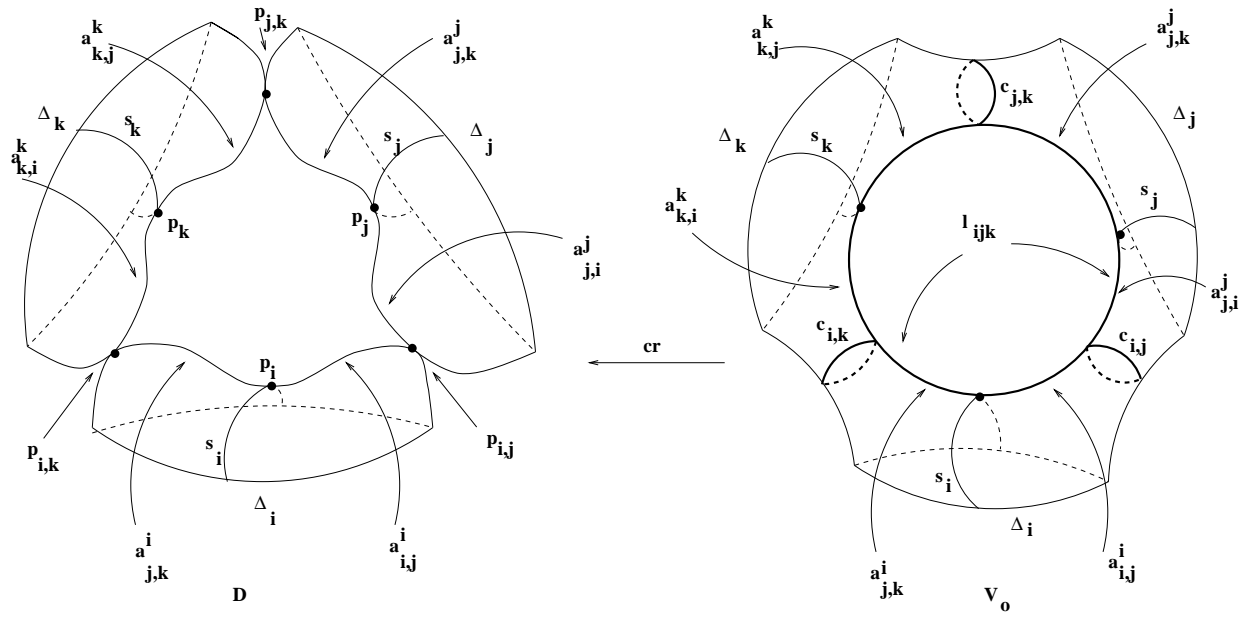


Figure 3.5: Deformation retraction of  $V_o$  onto  $D$ .



For each triple of distinct indices  $i, j, k$  denote by  $l_{i,j,k}$  the smooth unoriented circle in  $V_o$  obtained by tracing the segments  $a_{x,y}^x$ ,  $\{x, y\} \subset \{i, j, k\}$  in the following order (see Figure 3.5)

$$l_{i,j,k} = a_{k,i}^k \circ \bar{a}_{k,j}^k \circ a_{j,k}^j \circ \bar{a}_{j,i}^j \circ a_{i,j}^i \circ \bar{a}_{i,k}^i,$$

where we have adopted the convention, that  $a_{xy}^x$  is oriented from  $p_x$  to  $p_{x,y}$  and  $\bar{a}_{xy}^x$  is oriented from  $p_{x,y}$  to  $p_x$ .

By construction, the  $l_{ijk}$ 's are free loops representing elements in the kernel of the natural surjection  $(p_{\Sigma|V_o})_* : \pi_1(V_o) \rightarrow \pi_1(\Sigma)$ .

Before we state the main result of this section we need to introduce some notation. Let  $h$  be the genus of  $V_o$  and let  $S \subset \pi_1(V_o)$  be the set of geometric vanishing cycles for the algebraic pencil  $v$ . We have the following:

**Proposition 3.10** *Let  $L$  be the set of circles in minimal position on  $C_h$  defined as*

$$L := (\cup_i R_i) \cup S \cup \{l_{123}, l_{456}, \dots\}.$$

*Then there exists a SLF  $f : X \rightarrow S^2$  of genus  $h$  such that*

- *The set of vanishing cycles of  $f$  is exactly  $L$ .*
- *The fundamental group of  $X$  is canonically isomorphic to  $\Gamma$ .*
- *The identifications  $\pi_1(X) \cong \Gamma$  and  $V_o \cong C_h$  can be chosen so that the natural epimorphism  $\pi_h \rightarrow \pi_1(X)$  (induced from the inclusion of  $C_h$  in  $X$  as a regular fiber of  $f$ ) becomes the composition*

$$\pi_h = \pi_1(V_o) \xrightarrow{(p_{\Sigma|V_o})_*} \pi_1(\Sigma) = \pi_e \xrightarrow{\psi} \Gamma$$

**Proof.** Let  $\iota : V_o \hookrightarrow V$  be the natural inclusion of the divisor  $V_o$  in the surface  $V$ . Consider the induced map  $\iota_* : \pi_1(V_o) \rightarrow \pi_1(X)$ . Since  $V_o$  is the regular fiber of the algebraic Lefschetz fibration  $v : \widehat{V} \rightarrow \mathbb{P}^1$  we conclude by Lemma 3.2 (c) that  $\ker(\iota_*)$  is the normal subgroup generated by the circles in  $S$ . On the other hand since  $p_{\Sigma*}$  identifies  $\pi_1(V)$  with  $\pi_1(\Sigma)$  we have that  $\ker(\iota_*) = \ker((p_{\Sigma|V_o})_*)$  and so  $l_{ijk} \in \ker(\iota_*)$  for all triples of indices  $i, j, k$ . Due to this observation the normal subgroup in  $\pi_1(V_o) = \pi_h$  generated by the elements in  $L$  actually coincides with the normal subgroup generated by the elements in  $(\cup_i R_i) \cup S$ . Moreover by the handle body decomposition for the Lefschetz pencil  $v$  we have a commutative diagram

$$\begin{array}{ccc} \pi_1(V_o) & \xrightarrow{(p_{\Sigma|V_o})_*} & \pi_1(\Sigma) \\ & \searrow c\tau_* & \nearrow (p_{\Sigma|D})_* \\ & \pi_1(D) & \end{array}$$

and hence each set  $R_i \subset \pi_1(V_o) = \pi_h$  is mapped bijectively to the set  $R \subset \pi_e$  by the map  $\iota_*$ . This shows that  $L \subset \ker(\psi \circ \iota_*)$  and that moreover  $L$  generates  $\ker(\psi \circ \iota_*)$  as a normal subgroup in  $\pi_h$ .

Therefore By Lemma 3.2 the proposition will be proven if we can find a positive relation among the elements in  $L$ .

The set of circles  $S$  is the set of vanishing cycles in an algebraic Lefschetz pencil and so we have a positive relation among the elements in  $S$  as in Lemma 3.2. Thus  $\text{Map}_S(-) = \text{Map}_S$  by the “if” part of Lemma 3.5.

Put  $R := \cup_i R_i \cup \{l_{123}, l_{456}, \dots\}$ . By the geometric presentation assumption we know that each  $R_i$  is graph connected. Also by construction  $l_{ijk}$  is graph connected to the circles  $s_i \in R_i$ ,  $s_j \in R_j$  and  $s_k \in R_k$  and to the vanishing cycles  $c_{i,j}, c_{j,k}, c_{i,k} \in S$ . Therefore  $R$  is graph connected to  $S$  and so by Lemma 3.6 we have  $\text{Map}_{R \cup S}(-) = \text{Map}_{R \cup S}$ . Now we can apply the “only if” part of Lemma 3.5 to the set  $L = R \cup S$  which concludes the proof of the proposition.  $\square$

Granted the previous proposition Theorem A now becomes a tautology:

**Proof of Theorem A.** Given a finite presentation  $a : \pi_g \rightarrow \Gamma$  of  $\Gamma$  first use Lemma 3.8 to obtain a geometric presentation  $\psi : \pi_e \rightarrow \Gamma$  and then apply Proposition 3.10 to obtain a SLF with fundamental group  $\Gamma$ .  $\square$

### 3.4 Symplectic fibrations over bases of higher genus

The construction explained in the two previous sections can be easily modified to produce examples of symplectic Lefschetz fibrations over surfaces of genus bigger than zero (see Remark 3.11 for an explanation of the terminology). Moreover in that case the algebraic geometric part of the construction becomes superfluous and the resulting fibrations have monodromy groups generated essentially by the Dehn twists with respect to the circles giving the relations in some geometric presentation. This phenomenon is illustrated in Theorem B which we prove below.

**Remark 3.11** Before we prove the theorem a terminological remark is in order. Note that the notion of a symplectic Lefschetz fibration introduced in Definition 2.1 makes sense without the assumptions that the base of the fibration is a sphere and that the total space is four dimensional. In fact the whole setup of Definition 2.1 easily generalizes to fibrations of even dimensional manifolds over a base of even dimension.

In view of this we will adopt a slightly different terminology for the remainder of this section. Namely a symplectic Lefschetz fibration will mean a map  $f : X \rightarrow C$  from a smooth 4-fold  $X$  to a smooth compact Riemann surface  $C$  which satisfies all the conditions (i)-(iv) from Definition 2.1 but with  $S^2$  replaced by  $C$ .

This abuse of terminology should not cause any confusion since symplectic Lefschetz fibrations like that will be considered only in this section.

We can now state the main result of this section

**Proposition 3.12** *Let  $\Gamma$  be a finitely presentable group with a given presentation  $a : \pi_g \twoheadrightarrow \Gamma$ . Then there exist a geometric presentation  $\pi_e \rightarrow \pi_g \xrightarrow{a} \Gamma$  and a symplectic Lefschetz fibration  $f : X \rightarrow C_k$  for some  $k \gg 0$  so that*

- *the regular fiber of  $f$  is of genus  $e$ ,*

- the fundamental group of  $X$  fits in a short exact sequence

$$1 \rightarrow \Gamma \rightarrow \pi_1(X) \rightarrow \pi_k \rightarrow 1$$

- the natural surjective map from the fundamental group of the fiber of  $f$  to  $\Gamma$  coincides with the composition  $\pi_e \rightarrow \pi_g \xrightarrow{a} \Gamma$ .

**Proof.** Let  $\psi : \pi_d \rightarrow \pi_g \rightarrow \Gamma$  be a geometric presentation as in Lemma 3.8. Let  $R$  be a generating set of circles for  $\ker(\psi)$  as in Definition 3.7. Even though the circles in  $R$  need not satisfy a positive relation in  $\text{Map}_d$  the subsemigroup  $\text{Map}_R(-)$  is very close to being the whole group  $\text{Map}_R$  as the following arguments show.

**Lemma 3.13** *The abelianization of  $\text{Map}_R$  is a cyclic group.*

**Proof.** Fix a reference element  $c \in R$ . The group  $\text{Map}_R$  is generated by the Dehn twists  $\{t_s\}_{s \in R}$ . Since by definition any two circles in  $R$  are graph connected we conclude as in the proof of Lemma 3.6 that all generators of  $\text{Map}_R$  are conjugate to each other. Therefore any character  $\text{Map}_R \rightarrow S^1$  is defined by its value on  $t_c$  and so the quotient  $\text{Map}_R / [\text{Map}_R, \text{Map}_R]$  is generated by  $t_c[\text{Map}_R, \text{Map}_R]$ . The lemma is proven.  $\square$

The previous lemma shows that if we can find an integer  $n$  such that  $t_s^n \in [\text{Map}_R, \text{Map}_R]$  for any  $s \in R$ , then the abelianization of  $\text{Map}_R$  is of order at most  $n$ . In fact the same statement holds for any subgroup  $G$  of the mapping class group  $G \subset \text{Map}_d$  which is contained in the normal closure of  $\text{Map}_R$  in  $\text{Map}_d$ .

For the next step in the proof we will need to assume that the set  $R$  is big enough. More precisely let  $r, s \in R$  be two adjacent circles on  $C_d$  and let  $C_{r,s} \subset C_d$  be the corresponding handle. Let  $p$  be the intersection point of  $r$  and  $s$ . By deleting a small disk  $D_p \subset C_d$  centered at  $p$  and gluing a handle  $C_p$  in its place as in the proof of Lemma 3.8 we obtain a new surface  $C_e$  of genus  $e = d + 1$ . The two open curves  $r, s \subset C_d \setminus D_p$  can be completed to smooth circles by adding arcs in  $C_{r,s}$  as shown on Figure 3.6. Furthermore, we can enlarge the set of relations  $R$  to a new set of circles  $S = \{q\} \cup R \cup \{a, b\}$  where  $q \subset C_{r,s}$  is a circle isotopic to  $r$  such that  $r \cap D_p = \emptyset$  and  $a, b$  are the standard generators of the fundamental group of  $C_p$  (see Figure 3.6).

By construction we have a geometric presentation  $\pi_e \rightarrow \Gamma$  whose kernel is generated as a normal subgroup by the elements of  $S$ . By Lemma 3.13 the abelianization of the group  $\text{Map}_S$  is cyclic. In fact since we have ensured that  $S$  is big enough we have the following

**Lemma 3.14** *The abelianization of  $\text{Map}_S$  is a finite cyclic group of order dividing 10.*

**Proof.** The union  $(C_{r,s} \setminus D_p) \cup C_p$  is a genus two handle on  $C_e$  and therefore the inclusion  $(C_{r,s} \setminus D_p) \cup C_p \subset C_e$  induces a natural homomorphism  $h : \text{Map}_{2,1} \rightarrow \text{Map}_S$ . Moreover by construction the subgroup  $h(\text{Map}_{2,1}) \subset \text{Map}_S$  contains a non-trivial Dehn twist - e.g. the twist along the circle  $a \subset C_e$ . Therefore we conclude as in the proof of Lemma 3.13 that the group  $\text{Map}_S$  is generated by conjugates of the element  $t_a \in h(\text{Map}_{2,1}) \subset \text{Map}_S$ .

On the other hand it is known (see e.g. [Wajnryb83]) that the group  $\text{Map}_{2,1}$  has a presentation with several relations of degree zero and one relation of degree ten. More precisely one has

$$\text{Map}_{2,1} = \left\langle t_1, \dots, t_5 \left| \begin{array}{l} t_i t_j t_i = t_j t_i t_j \text{ if } |i - j| = 1, \text{ } t_i t_j = t_j t_i \text{ if } |i - j| \neq 1 \\ \text{and } (t_1 t_2 t_3)^4 = t_5 (t_4 t_3 t_2 t_1^2 t_2 t_3 t_4)^{-1} t_5 t_4 t_3 t_2 t_1^2 t_2 t_3 t_4 \end{array} \right. \right\rangle$$

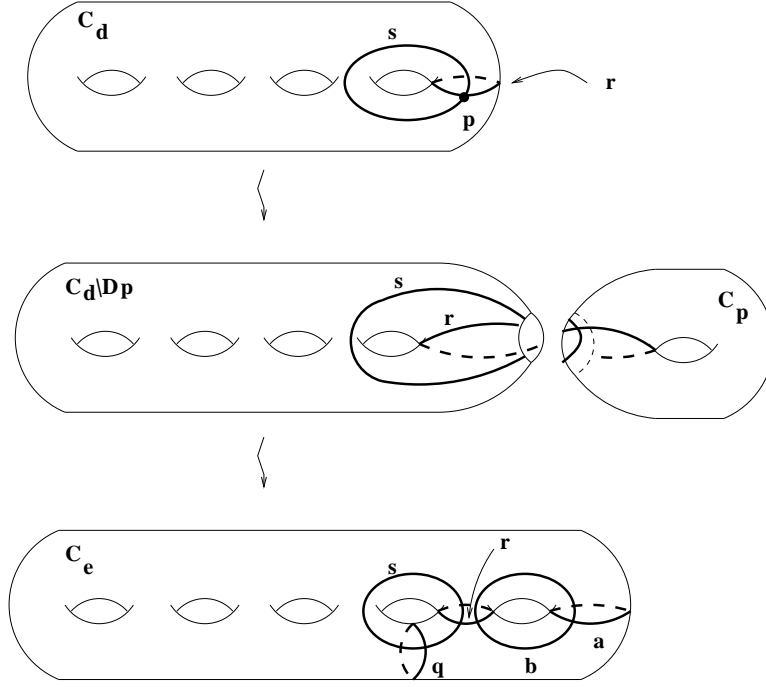


Figure 3.6: Enlarging a geometric presentation.

The braid relations  $t_i t_j t_i = t_j t_i t_j$  for  $|i - j| = 1$  in this presentation show that any homomorphism from  $\text{Map}_{2,1}$  to an abelian group  $G$  will have to map all  $t_i$ 's to the same element  $g \in G$ . Moreover the degree ten relation in the presentation implies that  $g^{10} = 1 \in G$  and so the abelianization of  $\text{Map}_{2,1}$  is isomorphic to  $\mathbb{Z}/10$ .

Due to this we see that the tenth power of any generator of  $\text{Map}_S$  will be conjugate to  $t_a^{10}$  and so will belong to the commutator subgroup  $[\text{Map}_S, \text{Map}_S]$ . Thus by the remark after the proof of Lemma 3.13 the abelianization of  $\text{Map}_S$  will be a cyclic group of order dividing ten. The lemma is proven  $\square$

Now we can finish the proof of the proposition. Let  $m$  be the cardinality of  $S$  and let  $s_1, \dots, s_m$  be an ordering of the elements in  $S$ . By Lemma 3.14 we can find elements  $\xi_i, \eta_i \in \text{Map}_S$ ,  $i = 1, \dots, k$  such that the relation

$$(3.1) \quad \prod_{j=1}^m t_{s_j}^{10} = \prod_{i=1}^k [\xi_i, \eta_i]$$

holds in  $\text{Map}_S$ .

For every  $i = 1, \dots, k$  choose diffeomorphisms  $\Xi_i, E_i \in \text{Diff}^+(C_e)$  representing the mapping classes  $\xi_i, \eta_i$ . Let  $K$  be a Riemann surface of genus  $k$  with one boundary component. The fundamental group  $\pi_1(K)$  is generated by  $2k + 1$  simple closed curves  $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, c$  satisfying the only relation

$$\prod_{i=1}^k [\alpha_i, \beta_i] = c$$

and so we can construct a representation  $\pi_1(K) \rightarrow \text{Diff}^+(C_e)$  given by  $\alpha_i \mapsto \Xi_i$ ,  $\beta_i \mapsto E_i$ ,  $c \mapsto \prod[\Xi_i, E_i]$ . Moreover since we have the freedom of changing the diffeomorphisms  $\Xi_i$  and  $E_i$  up to isotopy we may assume without a loss of generality that all the  $\Xi_i$ 's and  $E_i$ 's preserve a given symplectic form on  $C_e$ . With this choice the group  $\pi_1(K)$  acts symplectically and freely on the product  $\tilde{K} \times C_e$  of the universal cover of  $K$  and  $C_e$  and by passing to the quotient of this action we obtain a smooth symplectic fibration  $Y \rightarrow K$  with fiber  $C_e$ . The restriction of this fibration over the boundary circle  $c$  of  $K$  is a principal  $C_e$  fibration over  $S^1$  corresponding to the element  $\prod_i[\xi_i, \eta_i] \in \text{Map}_e$ . On the other hand exactly as in the proof of Lemma 3.2 we can construct a symplectic Lefschetz fibration  $u : U \rightarrow D$  whose restriction  $U|_{\partial D}$  to the boundary circle is a principal  $C_e$  fibration over  $S^1$  corresponding to the element  $\prod_j t_j^{10}$ . Due to the relation (3.1) this implies that the restrictions of  $Y$  and  $U$  to  $c \subset K$  and  $\partial D \subset D$  respectively are isomorphic  $C_e$  bundles over a circle and so we can glue  $Y$  and  $U$  along their boundaries to obtain a smooth orientable compact fourfold  $X$  which fibers over the genus  $k$  curve  $C_k = K \cup_{c=\partial D} D$  with a regular fiber of genus  $e$ . Since the embeddings  $U|_{\partial D} \subset U$  and  $Y|_c \subset Y$  have trivial normal bundles we can apply Weinstein's symplectic neighborhood theorem [Gompf95, Lemma 2.1] to patch the symplectic forms on  $U$  and  $Y$  over the gluing. Therefore the projection map  $f : X \rightarrow C_k$  is a symplectic Lefschetz fibration with  $10m$  singular fibers and vanishing cycles  $s_1, \dots, s_m$ . In particular  $\ker[\pi_1(X) \rightarrow \pi_1(C_k)]$  is isomorphic to the quotient of  $\pi_e$  by the normal subgroup generated by the  $s_j$ 's and so the proposition is proven.  $\square$

**Remark 3.15** It is clear from the proof of Proposition 3.12 that with a little extra care one can control effectively the genus of the fibers of  $f$  and the number of singular fibers of  $f$  but not the genus of the base  $C_k$ .

The trick of enlarging the geometric presentation by creating a genus two handle seems to be really necessary since a priori one can guarantee only the existence of a non-trivial homomorphism from  $\text{Map}_{1,1}$  to  $\text{Map}_d$  which is not enough since the abelianization of  $\text{Map}_{1,1}$  is  $\mathbb{Z}$  as explained in the proof of Corollary 5.2. On the other hand it is known that the group  $\text{Map}_{g,r}$  is perfect for  $g \geq 3$ ,  $r \geq 0$  [Powell78]. Hence same argument as in the proof of Proposition 3.12 shows that we can glue in an extra handle to  $C_e$  to obtain a non-trivial homomorphism  $\text{Map}_{3,1} \rightarrow \text{Map}_{d+2,1}$  and a geometric presentation  $\pi_{d+2} \rightarrow \Gamma$  whose kernel is generated as a normal subgroup by a connected graph of circles  $P$  with cardinality  $\#P = \#R + 5 = m + 2$ . In this case already the Dehn twists  $t_s$ ,  $s \in P$  themselves will be products of commutators in  $\text{Map}_P$  and so we will get a SLF with  $m + 2$  singular fibers.

In fact by further enlarging the genus of the base one can modify the fibration described in Remark 3.15 to obtain a proof of Theorem B. Actually we have the following precise version of Theorem B:

**Corollary 3.16** *Let  $\Gamma$  be a finitely presentable group with a given geometric presentation  $a : \pi_d \rightarrow \Gamma$ . Then there exist a geometric presentation  $\pi_{d+2} \rightarrow \pi_d \xrightarrow{a} \Gamma$  and a symplectic Lefschetz fibration  $f : Y \rightarrow C$  over a smooth compact Riemann surface  $C$  so that*

- *the regular fiber of  $f$  is diffeomorphic to  $C_{d+2}$ ,*
- *$f$  has a unique singular fiber,*

- the fundamental group of  $X$  fits in a short exact sequence

$$1 \rightarrow \Gamma \rightarrow \pi_1(X) \rightarrow \pi_1(C) \rightarrow 1$$

- the natural surjective map from the fundamental group of the fiber of  $f$  to  $\Gamma$  coincides with the composition  $\pi_{d+2} \rightarrow \pi_d \xrightarrow{a} \Gamma$

**Proof.** We will use the notation of Remark 3.15. As a first approximation to  $f : X \rightarrow C$  we will use the genus  $d + 2$  SLF constructed in Remark 3.15. Observe that since each  $t_s$ ,  $s \in P$  is a product of commutators in  $\text{Map}_P$  similarly to the proof of Theorem B we can replace a tubular neighborhood of the singular fiber at which the circle  $s$  vanishes with a smooth  $C_{d+2}$  fibration over a high genus Riemann surface corresponding to a representation of  $t_s$  as a product of commutators. If we do this for all elements in  $S$  but one we will get a symplectic Lefschetz fibration with one singular fiber and the same monodromy group as that of the original fibration. Let  $f : X \rightarrow C$  be this fibration and let  $q \in C$  be the only critical value of  $f$ . Let  $r \in P$  be the cycle vanishing at  $q$ . The map  $f$  induces a surjection on fundamental groups  $f_* : \pi_1(X) \rightarrow \pi_1(C)$  and the kernel  $\ker(f_*)$  is naturally a quotient of the fundamental group  $\pi_{d+2}$  of the regular fiber of  $f$ . To calculate  $\ker(f_*)$  note that we have a natural surjection  $\pi_1(X \setminus f^{-1}(q)) \rightarrow \pi_1(X)$  which restricts to the quotient map  $\pi_{d+2} \twoheadrightarrow \ker(f_*)$ . In particular  $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$  is precisely the subgroup in  $\pi_1(X \setminus f^{-1}(q))$  which is generated as normal subgroup (in  $\pi_1(X \setminus f^{-1}(q))$ ) by the cycle  $r \in \pi_{d+2} \subset \pi_1(X \setminus f^{-1}(q))$ . On the other hand the fact that  $X \setminus f^{-1}(q) \rightarrow C \setminus \{q\}$  is a smooth fibration identifies  $\pi_1(X \setminus f^{-1}(q))$  with the semi direct product  $\pi_1(C \setminus \{q\}) \ltimes_{\text{mon}} \pi_{d+2}$  where  $\text{mon} : \pi_1(C \setminus \{q\}) \rightarrow \text{Aut}(\pi_{d+2})$  is the monodromy representation. By the definition of a semi direct product the subgroup  $\pi_1(C \setminus \{q\}) \subset \pi_1(X \setminus f^{-1}(q))$  normalizes  $\pi_{d+2} \subset \pi_1(X \setminus f^{-1}(q))$  and the inner action of  $\pi_1(C \setminus \{q\})$  on  $\pi_{d+2}$  coincides with the representation  $\text{mon}$ . Consequently  $\ker[\pi_{d+2} \twoheadrightarrow \ker(f_*)]$  is the subgroup of  $\pi_{d+2}$  which is generated as a normal subgroup (in  $\pi_{d+2}$ ) by the orbit of the vanishing cycle  $c \in \pi_{d+2}$  under the monodromy group of  $f$ .

The next step is to recall that by construction the monodromy group of  $f$  is exactly  $\text{Map}_P$ . Moreover since  $P$  is a set of relations in a geometric presentation we know that the Dehn twists about any two circles in  $P$  are conjugate and hence for any  $s \in P$  there exists an element  $\phi \in \text{Map}_P$  for which  $t_s = \phi \circ t_r \circ \phi^{-1} = t_{\phi(r)}$ . However we have argued above that  $\phi(r) \in \ker(f_*)$  for any  $\phi \in \text{Map}_P$  and hence  $\ker f_*$  contains a circle which induces the same Dehn twist as  $s$ . To finish the proof we only need to observe that for any surface  $C_g$  and any circle  $c \subset C_g$  the Dehn twist  $t_c$  is uniquely characterized by the property that the maximal quotient of  $\pi_g$  on which it induces the identity is the quotient of  $\pi_g$  by the normal subgroup generated by  $c$ . Thus if we have two circles inducing the same Dehn twist the normal subgroups in  $\pi_g$  generated by those circles coincide. In combination with the above discussion this implies that  $Q \subset \ker(f_*)$  and so  $\ker(f_*)$  coincides with the quotient of  $\pi_{d+2}$  by the normal subgroup generated by the elements in  $Q$ . The corollary is proven.  $\square$

**Remark 3.17** Observe that even though there are no obvious restrictions for the Lefschetz fibrations constructed in Proposition 3.12 to be Kähler it will be very hard to construct projective examples like that. In general the algebraic Lefschetz fibrations have much bigger monodromies and as a result the fundamental group of an algebraic family  $f : X \rightarrow C$  as above is an extension of  $\pi_1(C)$  with a rather small (in general trivial) group. The reason for this phenomenon is that as it follows from the above argument the only condition for the existence of a symplectic Lefschetz

pencil with a given fundamental group is some equation in the mapping class group. On the other hand Hodge theory imposes many other restrictions in the algebraic situation - for example it is shown in [Katzarkov *et al.*98] that if the first Betti number of  $X$  is trivial, then the monodromy group of an algebraic Lefschetz pencil of big enough degree cannot fix any loop on the fiber. An interesting question to investigate is if this restriction exists for SLP.

## 4 Examples

In this section we describe some constructions of positive relations among right Dehn twists that do not use the general method employed in the proof of Theorem A but nevertheless lead to SLF with interesting fundamental groups.

### 4.1 Symplectic Lefschetz fibration with first Betti number equal to one

Our first construction produces a symplectic Lefschetz fibration  $f : X \rightarrow S^2$  with fibers of genus three having  $b_1(X) = 1$ . The first input for the construction is provided by the following lemma.

**Lemma 4.1** *The group  $\text{Map}_2$  contains five distinct right Dehn twists  $\{t_i\}_{i=1}^5 \subset \text{Map}_2$  satisfying the positive relation*

$$(t_1^2 \dots t_5^2)^2 = 1.$$

*Moreover the  $t_i$ 's can be chosen to be twists along non-separating circles in  $C_2$ .*

**Proof.** We will construct the Dehn twists  $t_i$  by exhibiting an algebraic family of genus two curves which has a single non-reduced fiber and local monodromies given by  $t_i^2$  at all other singular fibers.

To achieve this we start with the Hirzebruch surface  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$  with the two standard projections  $p_i : \mathbb{F}_0 \rightarrow \mathbb{P}^1$ ,  $i = 1, 2$ . Consider  $\mathbb{F}_0$  as a  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$  say via projecting onto the first factor. Choose six distinct points  $q, q_1, q_2, \dots, q_5 \in \mathbb{P}^1$  consider the divisors  $F = p_1^{-1}(q)$ ,  $H_i = p_2^{-1}(q_i)$ ,  $i = 1, \dots, 5$ . If we denote by  $\Delta \subset \mathbb{F}_0$  the diagonal divisor in  $\mathbb{P}^1 \times \mathbb{P}^1$ , then the divisor  $B := \sum_{i=1}^5 H_i + F + \Delta$  belongs to the linear system  $|\mathcal{O}_{\mathbb{F}_0}(2, 6)|$ . Since the line bundle  $\mathcal{O}_{\mathbb{F}_0}(2, 6)$  is divisible by two in  $\text{Pic}(\mathbb{F}_0)$  the divisor  $B$  determines a two sheeted root cover  $\alpha : S' \rightarrow \mathbb{F}_0$  branched at  $B$ . The divisor  $B$  has simple normal crossings and so the covering  $S'$  has an ordinary double point at each ramification point of  $\alpha$  sitting over a singularity of  $B$ . Denote by  $Q_i \in S'$  (respectively  $P_i \in S'$ ),  $i = 1, \dots, 5$  the ramification points of  $\alpha$  sitting over the intersection points of  $\Delta$  (respectively  $F$ ) with each of the  $H_i$ 's and let  $P_0 \in S'$  be the ramification point of  $\alpha$  projecting to the intersection point of  $\Delta$  and  $F$ .

Let  $\varepsilon : S \rightarrow S'$  be the minimal resolution of  $S'$  obtained by blowing up the  $P_j$ 's and the  $Q_i$ 's and let  $\mu_j = \varepsilon^{-1}(P_j)$ ,  $j = 0, \dots, 5$  and  $\nu_i := \varepsilon^{-1}(Q_i)$ ,  $i = 1, \dots, 5$  be the corresponding exceptional divisors. The composition  $p := p_1 \circ \alpha \circ \varepsilon : S \rightarrow \mathbb{P}^1$  is a flat morphism having as critical values exactly the points  $q, q_1, \dots, q_5$ . For each  $i = 1, \dots, 5$  the fiber  $p^{-1}(q_i)$  is reduced and is the union of an elliptic curve (the proper transform of  $(p_1 \circ \alpha)^{-1}(q_i)$ ) and a  $-2$  rational curve (the exceptional curve  $\nu_i$ ) meeting the elliptic curve transversally at two points. Therefore the geometric vanishing cycle at each of the points  $q_i$  is a system of two isotopic non-intersecting circles  $c'_i \cup c''_i \subset C_2$ . This implies that the geometric monodromy transformation around each  $q_i$  is given by the square of the

Dehn twist along  $c'_i$  (or equivalently along  $c''_i$ ). Therefore in  $\text{Map}_2$  we have a relation

$$t_1^2 \dots t_5^2 = \phi^{-1}$$

where  $\phi \in \text{Map}_2$  is the local geometric monodromy around the fiber  $p^{-1}(q)$ . Since  $F = p_1^{-1}(q)$  is part of the branch locus of the covering  $\alpha$  the scheme-theoretic fiber  $p^{-1}(q)$  will be nonreduced and will consist of one double rational component (whose reduction maps one to one onto  $F$ ) and six reduced rational components (the exceptional divisors  $\mu_j$ ). Finally observe that if we pull back (and normalize) the family  $p : S \rightarrow \mathbb{P}^1$  by a double cover  $D := \mathbb{P}^1 \rightarrow \mathbb{P}^1$  branched at  $q$  and at one extra point different from the  $q_i$ 's we will obtain a family over  $C$  with a smooth total space whose fiber over the point  $\tilde{q} \in C$  sitting over  $q$  is reduced and has one component of genus 2 (the double cover of  $F$  branched at the points  $P_j$ ,  $j = 0, \dots, 5$ ) and six components (sitting over the  $\mu_i$ 's) which are smooth rational curves of self-intersection  $-1$ . After contracting the six  $-1$  curves in the fiber we will obtain a genus two fibration which is smooth at  $q$  and hence has trivial monodromy at  $q$ . Thus the monodromy transformation  $\phi$  is of order two in  $\text{Map}_2$  and so the  $t_i$ 's satisfy

$$(t_1^2 \dots t_5^2)^2 = 1.$$

The lemma is proven. □

**Remark 4.2** (i) Clearly the construction used in the proof of Lemma 4.1 is very flexible and can be adapted to many different situations. One obvious generalization is the existence of  $2n + 1$  distinct Dehn twists in  $\text{Map}_n$  satisfying the relation

$$(t_1^2 \dots t_{2n+1}^2)^2 = 1.$$

(ii) Using the concrete nature of the construction in Lemma 4.1 one can obtain precise information about the homology classes and the position of the vanishing cycles  $c'_i$  and  $c''_i$  (see Lemma 4.3 below). In fact one can show that the free loops  $c'_i$ ,  $i = 1, \dots, 5$  generate the fundamental group  $\pi_2$  of the regular fiber of  $S$  and that the Dehn twists  $t_i$ ,  $i = 1, \dots, 5$  generate the mapping class group  $\text{Map}_2$ .

Before we state the second input to the construction we will need some preliminaries. We will use the notation introduced in the proof of Lemma 4.1. Fix a reference point  $o \in \mathbb{P}^1 \setminus \{q_0, q_1, \dots, q_5\}$  and let  $S_o \cong C_2$  be the corresponding fiber of  $S$ . Denote by  $v_i = [c'_i] = [c''_i] \in H_1(S_o, \mathbb{Z})$ ,  $i = 1, \dots, 5$  the homology classes of the vanishing cycles. Now we have

**Lemma 4.3** *There exists a character  $\chi : H_1(S_o, \mathbb{Z}) \rightarrow \mathbb{Z}/2$  which is uniquely determined by the property  $\chi(v_1) = 0$  and  $\chi(v_i) = 1$  for  $i \neq 1$ .*

**Proof.** Since the rank of the free abelian group  $H_1(S_o, \mathbb{Z})$  is four the existence of  $\chi$  will be proven if we can show that  $v_1, v_2, v_3, v_4$  generate  $H_1(S_o, \mathbb{Z})$ .

Note that by construction the family of hyperelliptic curves  $p : S \rightarrow \mathbb{P}^1$  is isotrivial and that the corresponding covering of Weierstrass points splits into six components (the strict transforms of the preimages of the divisors  $H_i$  and the divisor  $\Delta$ ). Therefore in each pair  $\{c'_i, c''_i\}$ ,  $i = 1, \dots, 5$  the vanishing cycles  $c'_i$  and  $c''_i$  get interchanged by the hyperelliptic involution and the image of the curve



$c'_i \cup c''_i$  via the projection  $g := p_2 \circ \alpha \circ \varepsilon : S \rightarrow \mathbb{P}^1$  is a simple closed curve  $\gamma_i \subset \mathbb{P}^1 \setminus \{q_0, q_1, \dots, q_5\}$  with the property that  $\mathbb{P}^1 \setminus \gamma_i$  is a union of two open disks one of which contains  $\{q_0, q_i\}$  and the other contains  $\{q_j\}_{j \neq 0, i}$ .

Now using the fact that  $g|_{S_o} : S_o \rightarrow \mathbb{P}^1$  is the hyperelliptic covering we can identify explicitly the homology classes  $v_i$ ,  $i = 1, \dots, 5$  as follows. Choose three non-intersecting segments  $\text{seg}_{01}, \text{seg}_{23}, \text{seg}_{45}$  in  $\mathbb{P}^1 \setminus \{q_0, q_1, \dots, q_5\}$  with endpoints  $\{q_0, q_1\}$ ,  $\{q_2, q_3\}$  and  $\{q_4, q_5\}$  respectively. As usual the hyperelliptic involution allows one to identify the genus two curve  $S_o$  with the topological surface which is obtained by first cutting  $\mathbb{P}^1$  along the segments  $\text{seg}_{01}, \text{seg}_{23}, \text{seg}_{45}$  and then gluing two copies of the resulting surface along the opposite shores of the cuts (see Figure 4.1).

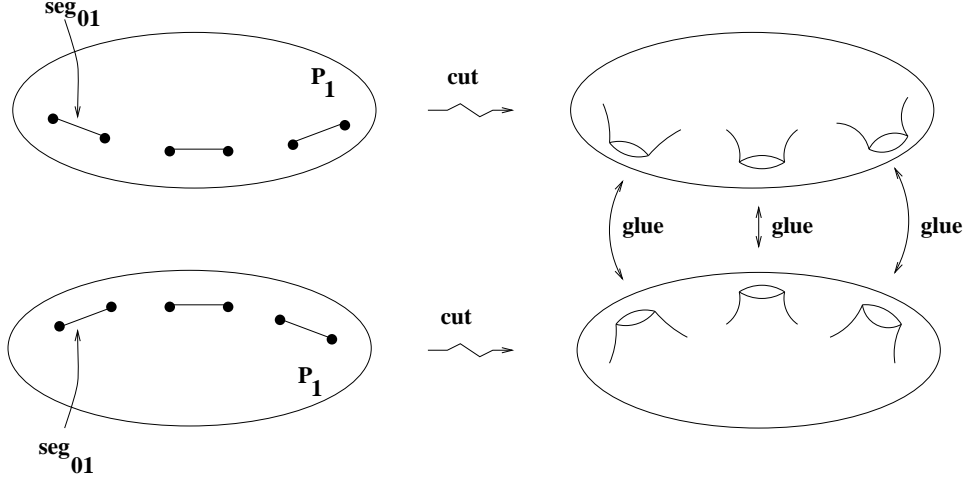


Figure 4.1: Gluing two  $\mathbb{P}^1$ 's into a hyperelliptic curve.

This explicit topological model of  $S_o$  together with the above characterization of the loops  $\gamma_i$  shows that the classes  $v_i$ ,  $i = 1, \dots, 5$  are represented by the loops shown on Figure 4.2.

Let now  $a_1, b_1, a_2, b_2$  be the standard basis of  $H_1(S_o, \mathbb{Z})$  which via the hyperelliptic map  $g$  projects to the loops  $\alpha_1, \beta_1, \alpha_2, \beta_2$  as depicted on Figure 4.3.

Using the explicit realization of the  $v_i$ 's one calculates  $v_1 = a_1$ ,  $v_2 = a_1 - b_1$ ,  $v_3 = -a_1 - b_1 + a_2$ ,  $v_4 = -b_1 - a_2 + b_2$ ,  $v_5 = b_1 + b_2$ . For this calculation we have used the standard orientations of the basis elements shown on Figure 4.3 and suitable orientations of the  $v_i$ 's. Note that since we are ultimately interested in the homology of  $S_o$  only modulo 2 the ambiguity introduced by the choice of orientations of the individual  $v_i$ 's will not affect anything.

The above formulas show that the desired character  $\chi$  is just the character for which  $a_1 \mapsto 1, b_1 \mapsto 0, a_2 \mapsto 1, b_2 \mapsto 1$ . The lemma is proven.  $\square$

Now we move to the construction of a family of genus 3 curves  $X$  with  $b_1(X) = 1$ .

Let  $C_2$  be a smooth surface of genus two and let  $c'_1, \dots, c'_5 \subset C_2$  be circles giving the Dehn twists  $t_1, \dots, t_5$  constructed in Lemma 4.1. Let  $d : D \rightarrow C_2$  be the unique unramified double cover

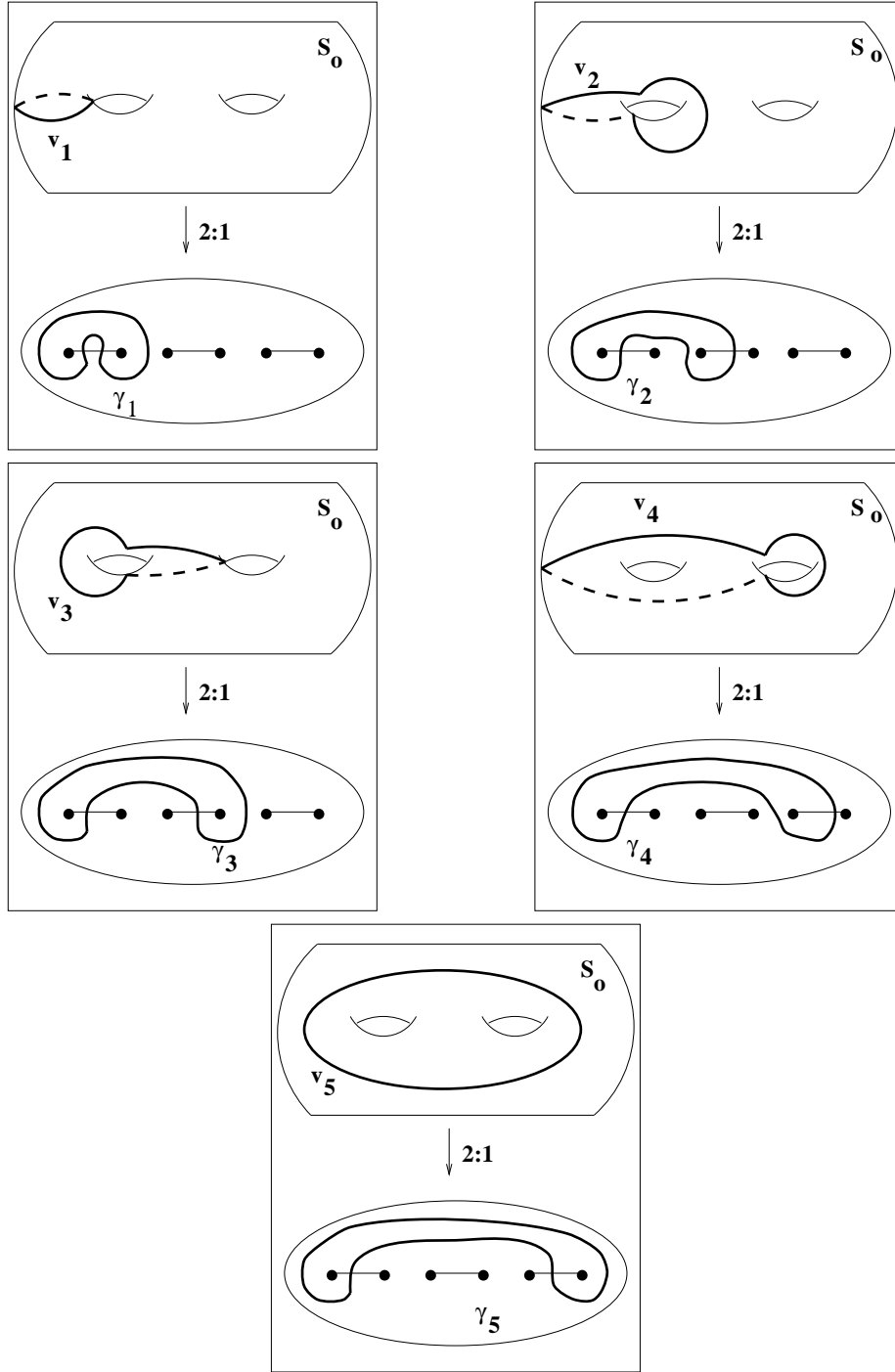


Figure 4.2: Loops representing the homology classes  $\{v_i\}_{i=1}^5$ .

of  $C_2$  corresponding to the character  $\chi$  from Lemma 4.3. By the definition of  $\chi$  we know that  $d^{-1}(c'_i) = w_i$  is connected for  $i \neq 1$  and  $d^{-1}(c_1) = u \cup v$  is a union of two non-intersecting circles

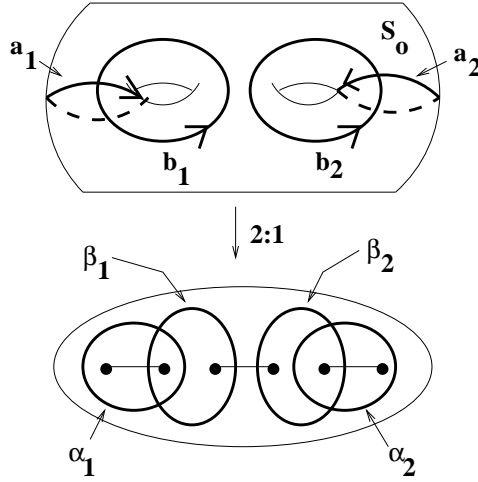


Figure 4.3: A standard basis in  $H_1(S_0, \mathbb{Z})$ .

which get interchanged by the involution on  $D$ . Therefore in  $\text{Map}_3$  we get a relation

$$(t_u t_v t_{w_2} t_{w_3} t_{w_4} t_{w_5})^2 = 1$$

Let  $f : X \rightarrow S^2$  be the symplectic Lefschetz fibration determined by this relation as in Lemma 3.2.

The homology classes of the  $w_i$  clearly generate the rank four subspace of  $H_1(D, \mathbb{Q})$  consisting of cycles invariant under the covering involution for the cover  $d : D \rightarrow C_2$ . Furthermore by construction the cycle  $u$  is not in the span of the  $w_i$ 's but  $u + v$  is invariant under the involution and so  $w_1, w_2, w_4, w_5, u, v$  span a five dimensional subspace in  $H_1(D, \mathbb{Q})$ . Since by Lemma 3.2 (c) we know that  $H_1(X, \mathbb{Q})$  is the quotient of  $H_1(D, \mathbb{Q})$  by the linear span of the vanishing cycles we conclude that  $b_1(X) = 1$  as desired.

**Remark 4.4** The family  $f : X \rightarrow S^2$  just constructed is an example of a symplectic Lefschetz fibration whose monodromy group does not act semi simply on the first cohomology of the fiber. This can be seen directly from the explicit description of the Dehn twists  $t_{w_i}$ ,  $t_u$  and  $t_v$  given above. In fact it is proven in [Katzarkov *et al.*98] that as long as the total space of a SLF has an odd first Betti number the monodromy action on the first cohomology of the fiber has a non-trivial unipotent radical.

**Remark 4.5** Explicit examples of symplectic Lefschetz fibrations with prescribed first homology were also constructed by Ivan Smith [Smith98]. In particular he shows that any abelian group which is a quotient of a free group on  $g$  generators can be realized as the first homology of a SLF with fiber genus  $2g$ .

## 4.2 Symplectic Lefschetz fibration with fundamental group equal to $\mathbb{Z}$ .

In this section we will use a slight modification of the proof of Proposition 3.10 to construct an example of a symplectic Lefschetz fibration with fundamental group equal to  $\mathbb{Z}$ .

Let  $E$  be an elliptic curve. Consider the complex projective surface  $V := E \times \mathbb{P}^1$  and let  $p_E : V \rightarrow E$  be the natural projection. Let  $D_\nu \subset V$  be the section of  $p_E$  given by the graph of the natural degree two covering  $\nu : E \rightarrow \mathbb{P}^1$ . Fix a point  $p \in \mathbb{P}^1$  which is not a branch point for the covering  $\nu : E \rightarrow \mathbb{P}^1$  and let  $D_p = E \times \{p\}$  be the corresponding section of  $p_E$ .

Consider a divisor  $D = D_p + D_\nu$ . By construction  $D$  is a strict normal crossing curve with exactly two singular points  $\{p_1, p_2\} = D_p \cap D_\nu$ . Suppose that the linear system  $|D|$  contains a smooth divisor  $Y$ . We will assume that  $Y$  is close enough to  $D$  in  $|D|$  so that there is a well defined deformation retraction  $cr : Y \rightarrow D$ . The curve  $Y$  is of genus 3 and the natural map  $p_{E|Y} : Y \rightarrow E$  is a two sheeted covering branched at four points which get glued in pairs to the points  $p_E(p_1)$  and  $p_E(p_2)$  respectively when we deform  $Y$  to  $D$ . Let  $\text{seg}_{12}$  be a contractible segment in  $E$  connecting  $p_E(p_1)$  and  $p_E(p_2)$  and let  $s \subset E$  be a non separating circle intersecting  $\text{seg}_{12}$  at a single point (see Figure 4.4). Denote by  $S$  the set of geometric vanishing cycles for an algebraic pencil  $\mathbb{P}^1 \subset |D|$  containing both  $Y$  and  $D$ . As in the proof of Proposition 3.10 consider the system of circles in minimal position

$$L := \{l_{12}, s_p, s_\nu\} \cup S \subset Y$$

where  $l_{12} := p_{E|Y}^{-1}(\text{seg}_{12})$  is the double cover of  $\text{seg}_{12}$  and  $s_p$  and  $s_\nu$  are the preimages of  $s$  in  $E_p$  and  $E_\nu$  respectively, viewed as cycles on  $Y$  via  $cr$  (see Figure 4.4). By definition the set  $S$  contains two cycles  $v_1$  and  $v_2$  corresponding to  $p_1$  and  $p_2$  respectively (i.e. the cycles contracted by  $cr$ ) and so  $L$  is graph connected.

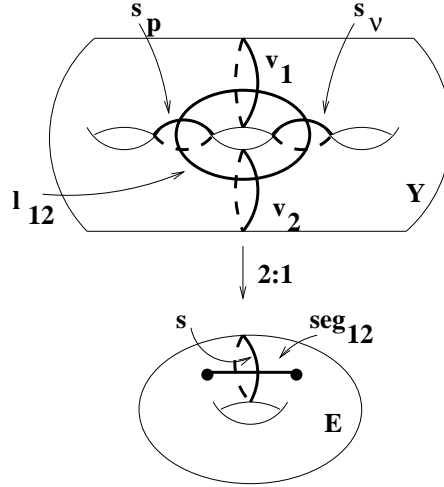


Figure 4.4: The covering  $p_{E|Y} : Y \rightarrow E$ .

The same argument as in the proof of Proposition 3.10 shows that the quotient  $\Gamma$  of  $\pi_1(Y)$  by the normal subgroup generated by the circles in  $L$  is isomorphic to the quotient of  $\pi_1(E)$  by the normal subgroup generated by  $s$ , i.e. is isomorphic to  $\mathbb{Z}$ . Furthermore since  $L$  is graph connected the combination of Lemma 3.2 and the “only if” part of Lemma 3.5 a SLF with monodromy group  $\text{Map}_L$ , regular fiber isomorphic to  $Y$  and fundamental group  $\Gamma$ .

Therefore in order to finish the construction we only need to show that the linear system  $|D|$  contains a smooth divisor. Let  $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$  and let  $a := \nu \times \text{id} : V \rightarrow \mathbb{F}_0$  be the double cover induced from  $\nu$ . As usually  $\mathcal{O}_{\mathbb{F}_0}(m, n)$  will denote the line bundle  $p_1^* \mathcal{O}_{\mathbb{P}^1}(m) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(n)$  where  $p_i : \mathbb{F}_0 \rightarrow \mathbb{P}^1$ ,  $i = 1, 2$  are the projections on the first and second factors respectively. By definition we have  $D_p \in |a^* \mathcal{O}_{\mathbb{F}_0}(0, 1)|$  and  $D_\nu \in |a^* \mathcal{O}_{\mathbb{F}_0}(1, 1)|$ . Thus  $D \in |a^* \mathcal{O}_{\mathbb{F}_0}(1, 2)|$  and so we need to find a smooth divisor in the linear system  $|a^* \mathcal{O}_{\mathbb{F}_0}(1, 2)|$ . But the double cover  $a : V \rightarrow \mathbb{F}_0$  is a root cover branched along a section of  $\mathcal{O}_{\mathbb{F}_0}(4, 0)$  and hence  $a_* \mathcal{O}_V = \mathcal{O} \oplus \mathcal{O}(-2, 0)$ . This yields

$$\begin{aligned} H^0(V, a^* \mathcal{O}_{\mathbb{F}_0}(1, 2)) &= H^0(\mathbb{F}_0, a_* a^* \mathcal{O}_{\mathbb{F}_0}(1, 2)) \\ &= H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1, 2) \otimes (\mathcal{O}_{\mathbb{F}_0} \oplus \mathcal{O}_{\mathbb{F}_0}(-2, 0))) \\ &= H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1, 2)) \end{aligned}$$

and so every member of the linear system  $|D|$  is a pullback of a divisor in  $|\mathcal{O}_{\mathbb{F}_0}(1, 2)|$ . Finally since  $\mathcal{O}_{\mathbb{F}_0}(1, 2)$  is obviously base point free the general point in the five dimensional projective space  $|\mathcal{O}_{\mathbb{F}_0}(1, 2)|$  will represent a smooth rational curve on  $\mathbb{F}_0$  which is a double cover of  $\mathbb{P}^1$  via the projection  $p_1$  and which intersects the branch divisor of  $a$  in eight distinct points. Therefore the preimage of such a curve in  $V$  is a smooth curve  $Y$  of genus three which finishes the construction.

**Remark 4.6** As an alternative to the previous construction one can use our main theorem in the following way. Start with a finitely presentable group  $\Gamma$  and let  $a_1 : \pi_g \twoheadrightarrow \Gamma$  and  $a_2 : \pi_g \twoheadrightarrow \Gamma$  be two presentations. Use Theorem A to construct two SLF  $f_1 : X_1 \rightarrow S^2$  and  $f_2 : X_2 \rightarrow S^2$  of the same genus  $h > g$  so that  $\pi_1(X_i) = \Gamma$  and the natural maps from the fundamental groups of the general fibers of  $f_i$  onto  $\Gamma$  factor as  $\pi_h \rightarrow \pi_g \xrightarrow{a_i} \Gamma$  for  $i = 1, 2$ . If we now glue the two pencils  $X_1$  and  $X_2$  along a regular fiber we obtain a new SLF  $f : X \rightarrow S^2$  of genus  $h$  which by van Kampen's theorem has fundamental group  $\pi_g / (\ker(a_1) * \ker(a_2))$ . In particular by gluing two genus two pencils with fundamental groups  $\mathbb{Z}^2$  one can get a genus two Lefschetz pencil with fundamental group  $\mathbb{Z}$ . In fact this procedure is the baby version of a sophisticated technique employed by Ivan Smith [Smitha] who showed that every quotient of  $\mathbb{Z}^{\oplus 2}$  can be realized as the fundamental group of a symplectic Lefschetz fibration.

Similar series of very elegant examples of genus two SLF with fundamental groups  $\mathbb{Z} \oplus \mathbb{Z}/n$  was constructed in [Ozbagci-Stipsicz98].

**Remark 4.7** It was pointed out to us by Ron Stern that similarly to [Ozbagci-Stipsicz98] and [Smitha] the examples with a fundamental group  $\mathbb{Z}$  and  $b_1 = 1$  we construct above are all not even homotopic to complex surfaces. Indeed from the classification of complex surfaces it follows that if the above surfaces are complex they are either secondary Kodaira surfaces, class VII surfaces or elliptic surfaces. The fact that our examples are symplectic excludes the first two possibilities. The third possibility is ruled out by the observation that if this is an elliptic fibration then it should be over  $S^2$ . But then  $b_1 \neq 1$  as it follows from [Friedman-Morgan94, Section 2.2].

## 5 Sections in symplectic Lefschetz fibrations

In this section we study the relation between numerical properties of the sections in a symplectic Lefschetz fibration  $f : X \rightarrow S^2$  and the geometric monodromy of  $f$ .

## 5.1 Mapping class groups in genus one

First we recall some well known facts on the structure of the mapping class groups in genus one (see e.g. [Birman88, Gervais96]).

Since any translation on a torus is homotopic to the identity, the natural map  $\text{Map}_1^1 \rightarrow \text{Map}_1$  is an isomorphism. Furthermore  $\text{Map}_1^1 = SL(2, \mathbb{Z})$  and coincides with the group of linear automorphisms of the two dimensional torus. The group  $\text{Map}_{1,1}$  is naturally identified with the mapping class group of a two dimensional torus with one hole. As in Remark 3.1 the natural forgetful map  $\text{Map}_{1,1} \rightarrow \text{Map}_1^1$  realizes  $\text{Map}_{1,1}$  as a central extension

$$(5.2) \quad 0 \rightarrow \mathbb{Z} \rightarrow \text{Map}_{1,1} \rightarrow SL(2, \mathbb{Z}) \rightarrow 1,$$

where the kernel  $\mathbb{Z}$  is generated by the right twist  $t_c$  along the boundary circle  $c$  of the hole.

On the other hand the group  $\text{Map}_{1,1}$  admits a presentation

$$(5.3) \quad \text{Map}_{1,1} = \langle t_a, t_b | t_a t_b t_a = t_b t_a t_b \rangle,$$

with generators  $t_a$  and  $t_b$  corresponding to the Dehn twist along a standard symplectic basis of cycles  $H_1(C_1, \mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}b$  on the non-punctured torus  $C_1$ . In particular under the natural map  $\text{Map}_{1,1} \rightarrow SL(2, \mathbb{Z})$  we have

$$t_a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, t_b \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

From this is clear that the element  $(t_a t_b)^3$  is central and maps to  $-1 \in SL(2, \mathbb{Z})$ . Thus  $(t_a t_b)^6$  generates  $\mathbb{Z} = \ker[\text{Map}_{1,1} \rightarrow SL(2, \mathbb{Z})]$ , i.e.  $(t_a t_b)^6 = t_c$  (see also [Ivanov-McCarthy95, Theorem 4.3]).

The central extension (5.2) corresponds to an element  $\tau \in H^2(SL(2, \mathbb{Z}), \mathbb{Z})$ . Since  $SL(2, \mathbb{Z})$  can be identified with the fundamental group of the moduli stack  $\mathcal{M}_1^1$  of elliptic curves, the element  $\tau$  can be interpreted as the first Chern class of a line bundle on  $\mathcal{M}_1^1$ . In fact Mumford had shown [Mumford65, Main Theorem] that  $\tau$  generates  $\text{Pic}(\mathcal{M}_1^1) \cong \mathbb{Z}/12$  and that  $\tau$  corresponds to the line bundle on  $\mathcal{M}_1^1$  which to every flat family  $p : E \rightarrow S$  of elliptic curves associates the element  $\bar{\omega}_f \in \text{Pic}(S)$  where  $\bar{\omega}_f$  is the pullback of the relative dualizing sheaf of  $f$  via the zero section.

More algebraically the central extension (5.2) can be described as follows. Consider the universal covering  $\widetilde{SL}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})$ . Since  $\pi_1(SL(2, \mathbb{R})) \cong \mathbb{Z}$  is a central subgroup in  $\widetilde{SL}(2, \mathbb{R})$  we can take the preimage  $\widetilde{SL}(2, \mathbb{Z})$  of  $SL(2, \mathbb{Z})$  in  $\widetilde{SL}(2, \mathbb{R})$ . By construction there is a natural central extension

$$(5.4) \quad 0 \rightarrow \mathbb{Z} \rightarrow \widetilde{SL}(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}) \rightarrow 1$$

One has the following simple but somewhat tedious lemma:

**Lemma 5.1** (a) *The group  $\text{Map}_{1,1}$  is isomorphic to  $\widetilde{SL}(2, \mathbb{Z})$ .*

(b) *The elements  $t_a, t_b \in \text{Map}_{1,1}$  are conjugate in  $\text{Map}_{1,1}$ .*

As an immediate corollary we get

**Corollary 5.2** *The subsemigroup of  $\text{Map}_{1,1}$  generated by the conjugates of the element  $t_a$  does not contain the identity element.*

The proofs of these statements will be given in Sections 5.2 and 5.3 respectively. The proof of Lemma 5.1 is based on the standard description [Lion-Vergne80, Section 1.8-1.9] of the universal cover of  $SL(2, \mathbb{R})$  which we recall next.

## 5.2 Universal covers of symplectic groups

Let  $V$  be a two dimensional real vector space with basis  $p, q \in V$  and coordinate functions  $x, y \in V^\vee$ . The group  $SL(2, \mathbb{R})$  is naturally identified with the group of linear automorphisms of  $V$  preserving the standard symplectic form  $w := x \wedge y \in \wedge^2 V^\vee$ . Let  $\Lambda$  denote the Grassmanian of all Lagrangian subspaces in  $V$ . Since  $\dim_{\mathbb{R}}(V) = 2$  we have  $\Lambda = Gr(1, V) \cong \mathbb{RP}^1 \cong S^1$ . In fact any line  $\ell \subset V$  is uniquely determined by the angle  $(\theta \bmod \pi)$  where  $\theta$  is the angle  $\ell$  forms with the  $x$ -axis. If we identify  $V \cong \mathbb{C}$  via  $(x, y) \mapsto x + iy$ , then the identification  $u : \Lambda \xrightarrow{\sim} S^1$  is given explicitly as  $u(\ell) = e^{2i\theta}$ , where  $\ell = \mathbb{R}e^{i\theta}p$ . The symplectic group  $SL(2, \mathbb{R})$  acts on  $\Lambda$  and this action lifts to a well defined continuous action of  $\widetilde{SL}(2, \mathbb{R})$  on the universal cover  $\widetilde{\Lambda} \cong \mathbb{R}$  of  $\Lambda$  which essentially determines  $\widetilde{SL}(2, \mathbb{R})$ .

Explicitly the group  $\widetilde{SL}(2, \mathbb{R})$  can be described in terms of the  $SL(2, \mathbb{R})$  action on  $\Lambda$  by means of the Maslov index [Cappell *et al.*94, Lion-Vergne80]. Recall [Lion-Vergne80, Section 1.5] that for a real symplectic vector space  $(V, w)$  with a Lagrangian Grassmanian  $\Lambda$ , the corresponding Maslov index is the function  $\tau : \Lambda^3 \rightarrow \mathbb{Z}$  defined as follows. For each triple  $(\ell_1, \ell_2, \ell_3)$  of Lagrangian subspaces of  $V$  consider the quadratic form  $Q_{\ell_1 \ell_2 \ell_3}(x)$  on the vector space  $\ell_1 \oplus \ell_2 \oplus \ell_3$  given by

$$Q_{\ell_1 \ell_2 \ell_3}(x_1 \oplus x_2 \oplus x_3) = w(x_1, x_2) + w(x_2, x_3) + w(x_3, x_1).$$

Next put  $\tau(\ell_1, \ell_2, \ell_3) :=$  the signature of  $Q_{\ell_1 \ell_2 \ell_3}$ . By construction the function  $\tau$  is antisymmetric in the three arguments and is invariant under the diagonal action of  $Sp(V, w)$  on  $\Lambda^3$ . Moreover if  $\ell \in \Lambda$  is a fixed Lagrangian subspace of  $V$ , then the  $\mathbb{Z}$ -valued function

$$\begin{aligned} \tau_\ell : \quad Sp(V, w) \times Sp(V, w) &\longrightarrow \mathbb{Z} \\ (g, h) &\longrightarrow \tau(\ell, g\ell, gh\ell), \end{aligned}$$

satisfies (see [Lion-Vergne80, Lemma 1.6.13])

$$\tau_\ell(g_1 g_2, g_3) + \tau_\ell(g_1, g_2) = \tau_\ell(g_1, g_2 g_3) + \tau_\ell(g_2, g_3).$$

Thus  $\tau_\ell$  is a cocycle for the group  $Sp(V, w)$  with coefficients in the trivial module  $\mathbb{Z}$  and so determines a central extension

$$(5.5) \quad 0 \rightarrow \mathbb{Z} \rightarrow G_\ell \rightarrow Sp(V, w) \rightarrow 1.$$

As a set  $G_\ell = Sp(V, w) \times \mathbb{Z}$  with the group structure being given by

$$(5.6) \quad (g_1, n_1) \cdot (g_2, n_2) = (g_1 g_2, n_1 + n_2 + \tau_\ell(g_1, g_2)).$$

The group  $G_\ell$  acts naturally on the universal cover  $\widetilde{\Lambda}$  of  $\Lambda$ . It turns out [Lion-Vergne80, Section 1.9] that there is a unique topology on  $G_\ell$  for which this action becomes continuous, and so  $G_\ell$  has a natural structure of a Lie group. Furthermore the universal covering group  $\widetilde{SP}(V, w)$  of  $Sp(V, w)$  is naturally identified with the identity component of  $G_\ell$ . In fact there is a canonical character  $s : G_\ell \rightarrow S^1$  of order four so that  $\widetilde{SP}(V, w) = \ker(s)$ .

This description of  $\widetilde{SP}(V, w)$  can be made completely explicit for the case of a two dimensional symplectic vector space  $(V, w)$  considered above. Indeed the Maslov index map  $\tau$  admits a very concrete description in this case. The natural (counterclockwise) orientation of  $S^1 \subset \mathbb{R}^2$  gives a cyclic ordering for any triple of points in  $S^1$ . Let  $\ell_1, \ell_2, \ell_3 \in \Lambda$  be three lines in  $V$ . Then the Maslov index  $\tau(\ell_1, \ell_2, \ell_3)$  is given by

$$\tau(\ell_1, \ell_2, \ell_3) = \begin{cases} 0 & \text{if } \ell_1, \ell_2, \ell_3 \text{ are not all distinct} \\ 1 & \text{if } u(\ell_2) \text{ is between } u(\ell_1) \text{ and } u(\ell_3) \\ -1 & \text{if } u(\ell_2) \text{ is between } u(\ell_3) \text{ and } u(\ell_1) \end{cases}$$

Geometrically this just means that  $\tau(\ell_1, \ell_2, \ell_3) = 1$  if  $\ell_2$  is inside the angle (mod  $\pi$ ) formed by  $\ell_1$  and  $\ell_3$  and  $\tau(\ell_1, \ell_2, \ell_3) = -1$  if  $\ell_2$  is outside that angle.

Choose  $\ell := \mathbb{R}p$  and consider the central extension (5.5). The function  $s : G_\ell \rightarrow S^1$  defined by

$$s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, n\right) := \begin{cases} \operatorname{sgn}(c)i^{n+1} & \text{if } c \neq 0 \\ \operatorname{sgn}(a)i^n & \text{if } c = 0 \end{cases}$$

is a character of  $G_\ell$  and  $\widetilde{SL}(2, \mathbb{R}) = \ker(s)$ . In other words we have

$$(5.7) \quad \widetilde{SL}(2, \mathbb{R}) = \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, n \right) \in SL(2, \mathbb{R}) \times \mathbb{Z} \left| \begin{array}{l} \text{If } c = 0, \text{ then } n \text{ is even and } \operatorname{sgn}(a) = (-1)^{n/2} \\ \text{and if } c \neq 0, \text{ then } n \text{ is odd and } \operatorname{sgn}(c) = (-1)^{(n+1)/2} \end{array} \right. \right\}$$

with a group law given by the formula (5.6).

Using this explicit description we can prove Lemma 5.1

**Proof of Lemma 5.1.** Put

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since  $B = JAJ^{-1}$  it suffices to check that there exist lifts  $\tilde{A}, \tilde{B}, \tilde{J} \in \widetilde{SL}(2, \mathbb{R})$  of  $A, B$  and  $J$  respectively so that

- $\tilde{B} = \tilde{J}\tilde{A}\tilde{J}^{-1}$
- $\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}\tilde{B}$ .
- $(\tilde{A}\tilde{B})^6$  generates  $\mathbb{Z} \cong \ker[\widetilde{SL}(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z})]$ .

Let  $k \in \mathbb{Z}$ . In terms of the description (5.7) choose

$$\tilde{A}_k = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 4k \right) \quad \tilde{J} = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \quad \tilde{B}_k = \tilde{J}\tilde{A}_k\tilde{J}^{-1} = \left( \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, 4k+1 \right).$$

Using the group law (5.6) one calculates

$$\tilde{A}_k\tilde{B}_k = \left( \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, 8k+1 + \tau(\mathbb{R}p, \mathbb{R}p, \mathbb{R}q) \right) = \left( \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, 8k+1 \right).$$



Similarly we have

$$\tilde{A}_k \tilde{B}_k \tilde{A}_k = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k + 1 + \tau(\mathbb{R}p, \mathbb{R}q, \mathbb{R}q) \right) = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k + 1 \right)$$

and

$$\tilde{B}_k \tilde{A}_k \tilde{B}_k = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k + 2 + \tau(\mathbb{R}p, \mathbb{R}(p - q), \mathbb{R}q) \right) = \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 12k + 1 \right).$$

Thus  $\tilde{A}_k \tilde{B}_k \tilde{A}_k = \tilde{B}_k \tilde{A}_k \tilde{B}_k$  as required. Furthermore we have

$$(\tilde{A}_k \tilde{B}_k \tilde{A}_k)(\tilde{B}_k \tilde{A}_k \tilde{B}_k) = \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 24k + 2 + \tau(\mathbb{R}p, \mathbb{R}q, \mathbb{R}p) \right) = \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 24k + 2 \right)$$

and hence

$$(\tilde{A}_k \tilde{B}_k)^6 = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 48k + 4 \right).$$

In addition from the description (5.7) we see that

$$\ker[\widetilde{SL}(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z})] = \left\{ \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 4k \right) \mid k \in \mathbb{Z} \right\}$$

and so  $(\tilde{A}_k \tilde{B}_k)^6$  will generate  $\ker[\widetilde{SL}(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z})]$  if  $k = 0$ . The Lemma is proven.  $\square$

**Remark 5.3** From the description (5.7) it is clear that the element  $\tilde{A}_k$  is the most general lift of  $A$ . Since any two lifts of  $J$  differ by a central element in  $\widetilde{SL}(2, \mathbb{Z})$  it follows that the pair  $(\tilde{A}_k, \tilde{B}_k)$  used in the proof of Lemma 5.1 represents the most general way to lift  $A$  and  $B$  to elements in  $\widetilde{SL}(2, \mathbb{Z})$  so that they still remain conjugate. The choice of  $k = 0$  guaranteeing the identification of  $\text{Map}_{1,1}$  with  $\widetilde{SL}(2, \mathbb{Z})$  has the following simple geometric interpretation.

The natural action of  $SL(2, \mathbb{R})$  on the Lagrangian Grassmanian  $\Lambda \cong S^1$  lifts to a non-trivial action of the universal covering group  $\widetilde{SL}(2, \mathbb{R})$  on the universal cover  $\tilde{\Lambda} \cong \mathbb{R}$  of  $\Lambda$ . The group  $\mathbb{Z} = \ker[\widetilde{SL}(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R})]$  acts discretely on  $\tilde{\Lambda}$  with a fundamental domain isomorphic to an interval of length  $2\pi$  and the action of  $k \in \mathbb{Z}$  on  $\tilde{\Lambda}$  is by translation by  $2k\pi$ . In particular for a point  $\tilde{\ell} \in \tilde{\Lambda} = \mathbb{R}$  and an element  $\tilde{g} \in \widetilde{SL}(2, \mathbb{R})$  we can define the *displacement angle* of  $\tilde{g}$  at  $\tilde{\ell}$  as the real number  $(\tilde{g}\tilde{\ell} - \tilde{\ell}) \bmod 2\pi\mathbb{Z}$ . For example if we identify the universal cover  $\tilde{U}(1) \cong \mathbb{R}$  with a subgroup of  $\widetilde{SL}(2, \mathbb{R})$ , then the elements in  $\tilde{U}(1) \cong \mathbb{R}$  have displacement angles that are independent of the choice of the point  $\tilde{\ell}$  and thus act as translations on  $\tilde{\Lambda} \cong \mathbb{R}$ .

If now  $1 \neq g \in SL(2, \mathbb{R})$  is a nilpotent element, then  $g$  has a fixed point on  $\Lambda$  and hence among all lifts of  $g$  in  $\widetilde{SL}(2, \mathbb{R})$  there is a unique one having a periodic sequence of fixed points in  $\tilde{\Lambda}$  with period  $\pi$ .

The lifts  $\tilde{A}_0$  and  $\tilde{B}_0$  are characterized uniquely (see the explanation below) as the lifts of  $A$  and  $B$  having periodic sequences of fixed points in  $\tilde{\Lambda}$ . In fact, granted this characterization, one can easily prove Lemma 5.1. Indeed, the matrix  $AB \in SL(2, \mathbb{R})$  is conjugate to a rotation by  $-\pi/3$  and so the displacement angle of any element in  $\widetilde{SL}(2, \mathbb{R})$  lifting  $AB$  at any point in  $\tilde{\Lambda}$  will be exactly  $-\pi/3$ . On the other hand since  $\tilde{A}_0$  and  $\tilde{B}_0$  each have a sequence of fixed points which is periodic

with period  $\pi$  we see that the displacement angles of  $\tilde{A}_0$  and  $\tilde{B}_0$  can not be smaller than  $-\pi$ . Thus both  $\tilde{A}_0\tilde{B}_0\tilde{A}_0$  and  $\tilde{B}_0\tilde{A}_0\tilde{B}_0$  have displacement angles which are strictly bigger than  $-2\pi$  and since they lift the same element  $ABA = BAB \in SL(2, \mathbb{R})$  we must have  $\tilde{A}_0\tilde{B}_0\tilde{A}_0 = \tilde{B}_0\tilde{A}_0\tilde{B}_0$ .

To justify the characterization of  $\tilde{A}_0$  and  $\tilde{B}_0$  as lifts of  $A$  and  $B$  having fixed points one proceeds as follows. Note that similarly to (5.7) the space  $\tilde{\Lambda}$  can be identified [Lion-Vergne80, Section 1.9] as a set with

$$\tilde{\Lambda} = \{(\ell, k) \in \Lambda \times \mathbb{Z} \mid k \equiv 1 + \dim_{\mathbb{R}}(\ell \cap \ell_0) \pmod{2}\}.$$

Under this identification the action of  $(g, n) \in \widetilde{SL}(2, \mathbb{R})$  on  $\tilde{\Lambda}$  is given by  $(g, n) \cdot (\ell, k) = (g\ell, n + k + \tau(\ell_0, g\ell_0, g\ell))$  and we immediately see that  $\tilde{A}$  fixes each of the points  $\{(\mathbb{R}p, 2k)\}_{k \in \mathbb{Z}}$  and that  $\tilde{B}$  fixes each of the points  $\{(\mathbb{R}q, 2k + 1)\}_{k \in \mathbb{Z}}$ .

**Remark 5.4** Lemma 5.1 (a) easily implies Mumford's theorem asserting that  $\text{Pic}(\mathcal{M}_1^1)$  is a cyclic group of order 12. Indeed due to Lemma 5.1 (a) we only need to check that the group of central extensions  $H^2(SL(2, \mathbb{Z}), \mathbb{Z})$  is isomorphic to  $\mathbb{Z}/12$  and is generated by the class of (5.4). Recall that  $SL(2, \mathbb{Z}) = (\mathbb{Z}/6) *_{(\mathbb{Z}/2)} (\mathbb{Z}/4)$  where the cyclic subgroups  $\mathbb{Z}/6$  and  $\mathbb{Z}/4$  are generated by the matrices  $AB$  and  $ABA$  respectively. The commutator subgroup of  $SL(2, \mathbb{Z})$  is a free subgroup  $\Phi$  on two generators, namely  $\Phi$  is the subgroup generated by  $[B, A]$  and  $[B, A^{-1}]$ . Hence  $SL(2, \mathbb{Z})$  fits in a short exact sequence  $0 \rightarrow \Phi \rightarrow SL(2, \mathbb{Z}) \rightarrow \mathbb{Z}/12 \rightarrow 0$  and we have the Hochschild-Serre spectral sequence

$$(5.8) \quad E_2^{pq} := H^p(\mathbb{Z}/12, H^q(\Phi, \mathbb{Z})) \Rightarrow H^{p+q}(SL(2, \mathbb{Z}), \mathbb{Z})$$

abutting to the cohomology of  $SL(2, \mathbb{Z})$ . This is a first quadrant spectral sequence and since  $\Phi$  is of cohomological dimension one only the first two rows of (5.8) are non-trivial. In particular (5.8) degenerates in the  $E_3$ -term and so  $E_{\infty}^{02} = E_2^{02} = 0$ ,  $E_{\infty}^{11} = \ker d_2^{11}$ ,  $E_{\infty}^{20} = E_2^{20}$ . Also since  $\mathbb{Z}$  is the trivial  $SL(2, \mathbb{Z})$  module we have

$$\begin{aligned} E_2^{20} &= H^2(\mathbb{Z}/12, H^0(\Phi, \mathbb{Z})) = H^2(\mathbb{Z}/12, (\mathbb{Z})^{\Phi}) = H^2(\mathbb{Z}/12, \mathbb{Z}) \\ E_2^{11} &= H^1(\mathbb{Z}/12, H^1(\Phi, \mathbb{Z})) = H^1(\mathbb{Z}/12, \text{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})) \\ &= H^1(\mathbb{Z}/12, \mathbb{Z}^2) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/12, \mathbb{Z}^2) = 0. \end{aligned}$$

In particular we have an isomorphism  $H^2(SL(2, \mathbb{Z}), \mathbb{Z}) = H^2(SL(2, \mathbb{Z})/\Phi, \mathbb{Z}) = H^2(\mathbb{Z}/12, \mathbb{Z})$ . On the other hand the pullback of the canonical central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

via any homomorphism  $\mathbb{Z}/12 \rightarrow \mathbb{Q}/\mathbb{Z}$  gives an identification  $H^2(\mathbb{Z}/12, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/12, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/12$ .

Finally since both  $AB$  and  $ABA$  are conjugate to rotations of finite order we can find maximal compact subgroups  $U', U'' \subset SL(2, \mathbb{R})$  so that  $AB \in U'$  and  $ABA \in U''$ . Furthermore since the inclusions  $U' \subset SL(2, \mathbb{R})$  and  $U'' \subset SL(2, \mathbb{R})$  are homotopy equivalences and since  $U'$  and  $U''$  are both isomorphic to  $\mathbb{R}/\mathbb{Z}$  we see that the pull backs of the extension (5.4) by the inclusions  $\mathbb{Z}/6 \subset SL(2, \mathbb{Z})$  and  $\mathbb{Z}/4 \subset SL(2, \mathbb{Z})$  are just the standard extensions

$$0 \rightarrow \mathbb{Z} \xrightarrow{6} \mathbb{Z} \rightarrow \mathbb{Z}/6 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathbb{Z} \xrightarrow{4} \mathbb{Z} \rightarrow \mathbb{Z}/4 \rightarrow 0$$

respectively. In particular the extension (5.4) is the pullback of

$$0 \rightarrow \mathbb{Z} \xrightarrow{12} \mathbb{Z} \rightarrow \mathbb{Z}/12 \rightarrow 0$$

via the homomorphism  $SL(2, \mathbb{Z}) \cong (\mathbb{Z}/6) *_{(\mathbb{Z}/2)} (\mathbb{Z}/4) \rightarrow \mathbb{Z}/12$  and hence corresponds to the generator of  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/12, \mathbb{Q}/\mathbb{Z})$ .

### 5.3 Sections in elliptic symplectic fibrations

Now we apply the above considerations to the study of symplectic Lefschetz fibrations of genus one.

Consider a symplectic Lefschetz fibration  $f : X \rightarrow S^2$  of genus one which corresponds to a relation

$$(5.9) \quad t_1^{n_1} t_2^{n_2} \dots t_m^{n_m} = 1, \text{ with } n_i > 0 \text{ for all } i = 1, \dots, m.$$

in  $\text{Map}_1^1 = SL(2, \mathbb{Z})$  as in Lemma 3.2. To avoid pathologies we will assume that  $f$  has a continuous section and that at least one of the Dehn twists  $t_i$  is a twist along a non-separating cycle. In that case it is known [Moishezon77, Theorem 9] [Kas80, Theorem 2.4] that the relation determines  $f : X \rightarrow S^2$  up to a diffeomorphism. Let  $\sigma$  be a smooth section of  $f$ . We would like to find the relationship between  $\sigma^2$  and the numbers  $n_i$  (see Corollary 5.5).

The Dehn twists  $t_i$  correspond to a sequence of smooth circles  $s_1, \dots, s_m$  in the generic fiber  $X_o \cong C_1$  of  $f$ . Since the twisting diffeomorphisms  $T_{s_i}$  are non trivial only in small neighborhoods of  $s_i$  in  $C_1$  we may assume without loss of generality that there is point  $p \in X_o$  and a small disc  $p \in \Delta \subset X_o$  so that that all  $T_{s_i}$  act identically on  $\Delta$ . Recall (Lemma 3.2) that the fibration  $f$  was reconstructed from the relation (5.9) in two steps. We produce first a SLF  $u : U \rightarrow D$  over a disc  $D$  with a boundary monodromy transformation  $\prod_i T_{s_i}^{n_i} : C_1 \rightarrow C_1$ . Next, since by assumption  $\prod_i T_{s_i}^{n_i}$  is homotopic to the identity we can complete the resulting fibration into an elliptic fibration over  $S^2$  by gluing the trivial genus one family along the boundary of  $D$ .

Now we can construct one section where the computation of the square can be made explicitly. Indeed the fibration  $u : U \rightarrow D$  has a section  $D \times \{p\}$  by construction. Also the fiber bundle  $u|_{u^{-1}(\partial D)} : u^{-1}(\partial D) \rightarrow \partial D$  is homotopy equivalent to  $\partial D \times X_o$ . Since  $\text{Map}_1^1 = \text{Map}_1$  we may choose this homotopy equivalence so that it stabilizes the point  $p \in X_o$  and thus glue the section  $D \times \{p\}$  with the constant section trough  $p$  in the trivial family we are gluing to  $D$  in order to complete  $u$  to  $f$ . Let  $\sigma$  denote the resulting section.

As we have just seen the right twists  $t_i$  can be viewed as elements of the group  $\text{Map}_{1,1}$  which is the group of classes of maps stabilizing  $\Delta \subset X_o = C_1$ . Then from the handle body decomposition described in Lemma 3.2 we conclude that there is a non-negative integer  $n$  so that

$$\prod t_i^{n_i} = t_c^n$$

where  $c$  is the boundary circle of  $C_1 \setminus \Delta$ . In particular the degree of the normal bundle to  $\sigma$  in  $X$  is exactly  $-n$ .

As a corollary from this observation one can obtain a symplectic version of a well known result of L. Szpiro [Szpiro90] who proved that for any jacobian elliptic fibration (elliptic fibration with a holomorphic section) over  $\mathbb{P}^1$  with only multiplicative fibers we have an inequality between the

number  $D$  of singular fibers and the number  $N$  of irreducible components of the singular fibers. More precisely he showed that  $D \leq 6N$ .

Since in the monodromy group the multiplicative fiber with  $m$  components corresponds to the element  $t^m$  for some right Dehn twist  $t$  we see that the following Corollary is a straightforward generalization of Szpiro's result to the symplectic category.

**Corollary 5.5** *Assume that in  $\text{Map}_{1,1} = \widetilde{SL}(2, \mathbb{Z})$  we have a relation  $\prod_{i=1}^m t_i^{n_i} = t_c^n$  with  $n_i > 0$ . Then  $\sum n_i = 12n$  and  $m > 2n$ .*

**Proof.** From the standard presentation (5.3) of  $\text{Map}_{1,1}$  we see that there exists a unique homomorphism  $\chi : \text{Map}_{1,1} \rightarrow \mathbb{Z}$  characterized by the property that  $\chi(t_a) = 1$  and  $\chi(t_b) = 1$ . Alternatively by Remark 5.4 the pushout of the extension (5.2) via the multiplication map  $\text{mult}_{12} : \mathbb{Z} \rightarrow \mathbb{Z}$  is a split extension and so we can choose for  $\chi$  the composition of  $\text{mult}_{12}$  with a splitting of this extension. Since  $t_c = (t_a t_b)^6$  we have that  $\chi(t_c) = 12$  and that

$$\sum n_i = \chi\left(\prod_{i=1}^m t_i^{n_i}\right) = \chi(t_c^n) = 12n.$$

Furthermore as explained in Remark 5.3 the transformation  $t_c^n$  translates every point in  $\mathbb{R} = \widetilde{\Lambda}$  by  $-2n\pi$ . On the other hand  $t_i^{n_i}$  projects to a nilpotent transformation in  $SL(2, \mathbb{Z})$  and so has a displacement angle  $-\pi$  at any point in  $\widetilde{\Lambda}$ . Hence  $m \geq 2n$ .  $\square$

We are now in a position to prove Corollary 5.2

**Proof of Corollary 5.2.** If the subsemigroup in  $\widetilde{SL}(2, \mathbb{Z})$  generated by the conjugates of  $t_a$  contains the identity element, then we can find a relation of the form (5.9) in  $\text{Map}_{1,1}$ . Hence there exist a SLF  $f : X \rightarrow S^2$  where the section corresponding to the puncture  $p$  has a trivial tubular neighborhood and normal bundle. By the above calculation this yields  $0 = \sigma^2 = -\sum n_i$ , i.e.  $n_i = 0$  for all  $i$ . This proves the corollary.  $\square$

**Remark 5.6** It is very tempting to try to extend this purely group theoretic proof of Szpiro's result to the elliptic curves over number fields. It is expected that the analogue of Szpiro's inequality with any constant instead of 6 will lead to a solution of the ABC-conjecture.

## 5.4 Symplectic fibrations with fibers of higher genus

Some of the features of  $\text{Map}_{1,1}$  that allowed us to restrict the numerical properties of the sections in a genus one SLF carry over to the case of SLF of higher genus. Here we outline an approach to the study of the numerical behavior of the sections in a higher genus SLF.

The main ingredient in the discussion in the previous section was the existence of a natural action of  $\text{Map}_{1,1}$  on the universal cover  $\widetilde{\Lambda} = \mathbb{R}$  of the Lagrangian Grassmanian  $\Lambda$  and the notion of a displacement angle.

There is a similar action of  $\text{Map}_{g,1}$  which we proceed to describe. For more details the reader may wish to consult the excellent exposition of S. Morita [Morita93].

As explained in Remark 3.1 the group  $\text{Map}_{g,1}$  is a central extension

$$(5.10) \quad 0 \rightarrow \mathbb{Z} \rightarrow \text{Map}_{g,1} \rightarrow \text{Map}_g^1 \rightarrow 1$$

of  $\text{Map}_g^1$  by an infinite cyclic group generated by the Dehn twist around the puncture.

This central extension has many remarkable properties but we want to emphasize the ones which are reflected in the structure of the semi group  $\text{Map}_{g,1}(-) \subset \text{Map}_{g,1}$  generated by all right handed Dehn twists.

Let  $e \in H^2(\text{Map}_g^1, \mathbb{Z})$  denote the class of the central extension (5.10). The class  $e$  is dubbed the *Euler class* by Morita and admits the following simple description [Morita93]. As in the beginning of Section 3.1 we will view the elements in  $\text{Map}_g^1$  as isotopy classes of orientation preserving diffeomorphisms of  $C_g$  that preserve the point  $x_1 \in C_g$ .

Fix an isomorphism of the universal covering of  $C_g$  with the unit disk  $D \subset \mathbb{C}$ , e.g. by fixing a point  $j$  in the Teichmüller space  $\mathcal{T}_g$ . Let  $\nu : D \rightarrow C_g$  be the corresponding covering map. Any orientation preserving diffeomorphism  $\Phi : C_g \rightarrow C_g$  with  $\Phi(x_1) = x_1$  defines a quasi-conformal map  $h^\Phi : D \rightarrow D$  which preserves the preimage  $\nu^{-1}(x_1)$  of  $x_1$  in  $D$ . This map extends to an orientation preserving homeomorphism  $h_\partial^\Phi : S^1 \rightarrow S^1$  of the boundary  $\partial D \cong S^1$  of  $D$ . Modulo isometries of  $D$  the homeomorphism  $h_\partial^\Phi$  is uniquely defined by the isotopy class of  $\Phi$  relative to  $p$ . If in addition we require that the isomorphism  $j : \tilde{C}_g \xrightarrow{\sim} D$  sends a marked point  $\tilde{x}_1 \in \nu^{-1}(x_1)$  to  $0 \in D$  and the differential  $dj$  induces a fixed isomorphism of the tangent spaces  $T_{\tilde{x}_1} \tilde{C}_g$  and  $T_0 D$ , then the element  $h_\partial^\Phi \in \text{Homeo}^+(S^1)$  is uniquely determined by the mapping class  $[\Phi] \in \text{Map}_g^1$  of  $\Phi$ .

Therefore we obtain a natural action  $\rho : \text{Map}_g^1 \rightarrow \text{Homeo}^+(S^1)$  given by  $\rho([\Phi]) = h_\partial^\Phi$ . The group  $\text{Homeo}^+(S^1)$  has a natural central extension by an infinite cyclic group which is the group  $\text{Homeo}^{\text{per}}(\mathbb{R})$  of  $2\pi$ -periodic orientation preserving homeomorphisms of the line  $\mathbb{R}$  (here as before one should view the line  $\mathbb{R}$  as the universal cover of the circle  $S^1$ ). One can check [Morita88] that the central extension (5.10) is just the pullback of the extension  $0 \rightarrow \mathbb{Z} \rightarrow \text{Homeo}^{\text{per}}(\mathbb{R}) \rightarrow \text{Homeo}^+(S^1) \rightarrow 1$  by the homomorphism  $\rho$ . In particular we have a natural homomorphism  $\rho^{\text{per}} : \text{Map}_{g,1} \rightarrow \text{Homeo}^{\text{per}}(\mathbb{R})$  and so in the same way as in Remark 5.3 we can define the *displacement angle* associated with any pair  $(\phi, \ell)$ ,  $\phi \in \text{Map}_{g,1}$ ,  $\ell \in \mathbb{R}$  as the real number  $(\rho^{\text{per}}(\phi)(\ell) - \ell) \bmod 2\pi\mathbb{Z}$ .

We propose the following conjectural characterization of the semigroup  $\text{Map}_{g,1}(-)$  which generalizes Remark 5.3:

**Conjecture 5.7** *The subsemigroup  $\text{Map}_{g,1}(-) \subset \text{Map}_{g,1}$  consists of mapping classes whose displacement angle is non-negative at every point in  $\mathbb{R}$ .*

Before we give some evidence for the validity of this conjecture we need to recall Morita's analysis of the Euler class  $e$ .

Similarly to the genus one case we can view the Euler class  $e$  as an element of Picard group of the moduli space  $\mathcal{M}_g^1$  of smooth curves of genus  $g$  with one marked point. As such  $e$  can be written explicitly [Harris84], [Morita89] as a linear combination with  $\mathbb{Q}$ -coefficients of two cocycles of geometric origin:  $e_1$  and  $c$ . The cocycle  $e_1$  is just the first Mumford-Miller-Morita class and is proportional to the generator of the Picard group of the moduli space  $\mathcal{M}_g$  of smooth curves of genus  $g$ . Also as in the genus one case the cocycle  $e_1$  is induced from a cocycle for  $Sp(H_1(C_g, \mathbb{Z}))$  via the natural homomorphism  $\text{Map}_g^1 \rightarrow \text{Map}_g \rightarrow Sp(H_1(C_g, \mathbb{Z}))$ . The cocycle  $c$  is somewhat more mysterious. Geometrically it can be defined as follows. Let  $\Theta$  denote the pullback of the first Chern class of the relative theta line bundle on the universal degree  $g-1$  Jacobian  $\mathcal{J}^{g-1} \rightarrow \mathcal{M}_g$  via the canonical Abel-Jacobi map from the universal curve  $\mathcal{M}_g^1 \rightarrow \mathcal{M}_g$  to  $\mathcal{J}^{g-1} \rightarrow \mathcal{M}_g$  defined by the point  $x_1 : \mathcal{M}_g \rightarrow \mathcal{M}_g^1$ . Then it can be shown that  $c = 8\Theta - e_1/3$  [Morita89, (1.7)] and [Hain-Reed98, Proposition 2].

Observe that these geometric definitions of the classes  $e_1$  and  $c$  make sense in the genus one case as well. However it follows from the proof of Mumford's result [Mumford65, Main Theorem] discussed in Section 5.1 that in the genus one case the two classes  $e_1$  and  $c$  coincide and so we are not getting anything new for the group  $\text{Map}_1 = SL(2, \mathbb{Z})$ . In contrast for  $g > 2$  the classes  $e_1$  and  $c$  are linearly independent in  $H^2(\text{Map}_g^1, \mathbb{Q})$  and are related to the class  $e$  by the following explicit formula (see e.g. [Morita89])

$$e = \frac{1}{4g(1-g)}(e_1 + c).$$

Algebraically the classes  $e_1$  and  $c$  can be interpreted as follows. Consider the lattice  $H := H_1(C_g, \mathbb{Z})$  together with its natural symplectic form  $\theta$  given by the intersection pairing. Morita has shown (see e.g. [Morita88]) that there exists a two-step nilpotent group  $\mathcal{U}$  equipped with a  $Sp(H, \theta)$ -action and a homomorphism  $\kappa : \text{Map}_g^1 \rightarrow Sp(H, \theta) \ltimes (\mathcal{U} \otimes \mathbb{Q})$  so that the classes  $e_1$  and  $c$  are pullbacks of natural cohomology classes on  $Sp(H, \theta) \ltimes (\mathcal{U} \otimes \mathbb{Q})$ . More precisely  $\mathcal{U}$  is a  $Sp(H, \theta)$ -invariant central extension of  $\wedge^3 H$  by a certain  $Sp(H, \theta)$ -module of finite rank and  $e_1$  is a pullback of a cohomology class of  $Sp(H, \theta)$  and  $c$  is a pullback of a cohomology class  $e_1$  of  $Sp(H, \theta) \ltimes \frac{1}{2} \wedge^3 H$  via the natural maps

$$(5.11) \quad \text{Map}_g^1 \xrightarrow{\kappa} Sp(H, \theta) \ltimes (\mathcal{U} \otimes \mathbb{Q}) \rightarrow Sp(H, \theta)$$

$$(5.12) \quad \text{Map}_g^1 \xrightarrow{\kappa} Sp(H, \theta) \ltimes (\mathcal{U} \otimes \mathbb{Q}) \rightarrow Sp(H, \theta) \ltimes (\wedge^3 H \otimes \mathbb{Q})$$

respectively.

To identify the element in  $H^2(Sp(H, \theta), \mathbb{Z})$  that pulls back to  $e_1$  consider the  $2g$ -dimensional vector space  $H_{\mathbb{R}} := H \otimes \mathbb{R}$ . The symplectic group  $Sp(H_{\mathbb{R}}, \theta)$  is homotopy equivalent to its maximal compact subgroup which in turn is isomorphic to the unitary group  $U(g)$ . In particular  $\pi_1(Sp(H_{\mathbb{R}}, \theta)) \cong \pi_1(U(g)) \cong \mathbb{Z}$  and so the universal cover  $\widetilde{Sp}(H_{\mathbb{R}}, \theta)$  of  $Sp(H_{\mathbb{R}}, \theta)$  is naturally a central extension of  $Sp(H_{\mathbb{R}}, \theta)$  by an infinite cyclic group. By pulling back this extension by the natural inclusion  $Sp(H, \theta) \subset Sp(H_{\mathbb{R}}, \theta)$  we get a central extension

$$(5.13) \quad 0 \rightarrow \mathbb{Z} \rightarrow \widetilde{Sp}(H, \theta) \rightarrow Sp(H, \theta) \rightarrow 1.$$

From the geometric description of  $e_1$  given above it is clear that  $e_1$  is proportional to the pullback of the extension class (5.13)<sup>1</sup>.

To identify the element in  $H^2(Sp(H, \theta) \ltimes \frac{1}{2} \wedge^3 H, \mathbb{Z})$  that pulls back to  $c$  notice that the contraction with  $\theta$  gives a well defined homomorphism of  $Sp(H, \theta)$ -modules  $C : \wedge^3 H \rightarrow H$ . Consider the Heisenberg central extension  $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{H} \rightarrow H \rightarrow 0$  corresponding to the class  $\theta \in H^2(H, \mathbb{Z})$ . Since by definition  $\theta$  is  $Sp(H, \theta)$ -invariant the pullback of the Heisenberg extension via the map  $C$  will also be  $Sp(H, \theta)$ -invariant and will so determine an element  $H^2(Sp(H, \theta) \ltimes \frac{1}{2} \wedge^3 H, \mathbb{Z})$ . In [Morita88, Theorem 3.1] Morita shows that this element pulls back to  $c$  via the homomorphism (5.12).

It is not hard to see that the above interpretation of  $c$  gives rise to a representation of  $\text{Map}_{g,1}$  into the universal cover of a different symplectic group. Indeed, let us fix a symplectic basis of  $H$

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<sup>1</sup>In fact it was pointed out to us by R. Hain that the extension class (5.13) determines the universal central extension of  $Sp(H, \theta)$  and so with the correct choice of the orientation of the fiber must be equal to the pullback of the determinant of the pushforward of the relative cotangent bundle on the universal abelian variety or in other words to  $e_1/12$ .

and identify  $Sp(H, \theta)$  with the group  $Sp(2g, \mathbb{Z})$ . Denote by  $ASp(2g, \mathbb{Z})$  the subgroup of the group  $Sp(2g + 2, \mathbb{Z})$  which stabilizes the orthogonal complement of the element  $b_{g+1}$  of the standard symplectic basis  $a_1, b_1, \dots, a_{g+1}, b_{g+1}$  of  $\mathbb{Z}^{2g+2}$ . The natural map  $ASp(2g, \mathbb{Z}) \rightarrow GL(2g + 1, \mathbb{Z})$ ,  $X \mapsto X|_{b_{g+1}^\perp}$  maps the group  $ASp(2g, \mathbb{Z})$  onto  $Sp(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$  and so  $ASp(2g, \mathbb{Z})$  fits into a central extension

$$(5.14) \quad 0 \rightarrow \mathbb{Z} \rightarrow ASp(2g, \mathbb{Z}) \rightarrow Sp(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g} \rightarrow 1$$

which is clearly isomorphic with the central extension of  $Sp(H, \theta) \ltimes H$  considered above.

The fact that  $e$  is proportional to  $e_1 + c$  combined with the above discussion implies that the group  $\text{Map}_{g,1}$  has a well defined homomorphism  $\delta : \text{Map}_{g,1} \rightarrow \widetilde{ASp}(2g, \mathbb{Z})$  to the universal cover  $\widetilde{ASp}(2g, \mathbb{Z})$  of  $ASp(2g, \mathbb{Z})$ .

On the other hand it is clear from the construction that  $\widetilde{ASp}(2g, \mathbb{Z})$  is also isomorphic to the preimage of  $ASp(2g, \mathbb{Z})$  into the universal cover of  $Sp(2g + 2, \mathbb{R})$ . The choice of an inclusion  $U(g + 2) \subset Sp(2g + 2, \mathbb{R})$  induces an isomorphism of the Lagrangian Grassmanian  $\Lambda$  of  $\mathbb{R}^{2g+2}$  with the quotient  $U(g + 1)/O(2g + 2)$ . Since  $O(2g + 2) \subset SU(g + 1)$  we have a well defined map  $\det : \Lambda \rightarrow U(g + 1)/SU(g + 1) \cong S^1$  and in particular a well defined map of the universal covers  $\widetilde{\det} : \widetilde{\Lambda} \rightarrow \mathbb{R}$ . Since the group  $\widetilde{Sp}(2g + 2, \mathbb{R})$  acts naturally on  $\widetilde{\Lambda}$  we can use the map  $\widetilde{\det}$  to define a displacement angle for any element  $A \in \widetilde{Sp}(2g + 2, \mathbb{R})$  and any  $\ell \in \mathbb{R}$ .

Since the homomorphism  $\delta : \text{Map}_{g,1} \rightarrow \widetilde{ASp}(2g, \mathbb{Z})$  is essentially given by the class  $e$  it is reasonable to expect that for any  $\phi$  and any  $\ell \in \mathbb{R}$  the displacement angles of  $(\rho^{\text{per}}(\phi), \ell)$  and  $(\delta(\phi), \ell)$  will have the same sign. If this is the case, then the validity of Conjecture 5.7 will easily follow since similarly to the genus one case we can split the all unipotent elements in  $ASp(2g, \mathbb{Z})$  into two classes according to their displacement angle. That is - unipotent elements whose lifts in  $\widetilde{ASp}(2g, \mathbb{Z})$  have a periodic sequence of points with displacement angle zero and unipotent elements whose lifts in  $\widetilde{ASp}(2g, \mathbb{Z})$  do not have points with displacement angle zero.

Since the Dehn twists obviously belong to the first class and since the elements in the first class will generate the subsemigroup in  $\widetilde{ASp}(2g, \mathbb{Z})$  consisting of elements with non-positive displacement angle this argument will prove Conjecture 5.7.

Similar arguments should also lead to a proof of the following theorem, which is obviously correct in the case of projective Lefschetz pencils.

**Theorem 5.8 (I.Smith)** *There are no non-trivial SLF whose monodromy group is contained in the Torelli group.*

A special case of the above theorem was originally proven by B.Ozbagci [Ozbagci98, Corollary 7] who showed that hyperelliptic SLF (and in particular all Lefschetz fibrations of fiber genus two) cannot have monodromy contained in the Torelli group. The general statement of Theorem 5.8 appeared as a conjecture in a preliminary version of the present paper. The conjecture was settled affirmatively by Ivan Smith who graciously provided us with the proof appearing in Appendix A below.

In this direction we would like to ask a couple of questions:

Let  $\mathcal{L} \rightarrow \mathcal{M}_g$  be the Hodge line bundle and let  $c_1(\mathcal{L}) \in CH^1(\overline{\mathcal{M}}_g, \mathbb{Q})$  denote the natural extension of the first Chern class of  $\mathcal{L}$  to the Deligne-Mumford compactification of  $\mathcal{M}_g$ .

**Question 5.9** Let  $f : X \rightarrow C$  be a symplectic Lefschetz fibration of fiber genus  $g$  over an arbitrary Riemann surface  $C$ . Is it true that

$$\langle c_1(\mathcal{L}), [C] \rangle \geq 0?$$

This question has an affirmative answer (see Corollary A.6) whenever  $C$  is of genus zero.

**Question 5.10** Let  $f : X \rightarrow C$  be a symplectic Lefschetz fibration of fiber genus  $g$  over an arbitrary Riemann surface  $C$ . Is it true that

$$(8g + 4)\langle c_1(\mathcal{L}), [C] \rangle - g \cdot \mu \geq 0?$$

Here as usual  $\mu$  denotes the number of the singular fibers in the fibration  $f : X \rightarrow C$ .

The last question is a symplectic version of the Moriwaki inequality and was suggested by R. Hain.

If the answers of the above questions are positive we get additional restrictions on the words in the mapping class group defining SLF. Some analogues of Corollary 5.5 for high genus SLF can be expected as well.



## Appendix A (by Ivan Smith) Torelli fibrations

The purpose of this Appendix is to present a short proof of Theorem 5.8.

Suppose  $f : X \rightarrow S^2$  is a SLF whose monodromy is contained in the Torelli group. The SLF  $f$  induces a sphere  $S \cong S^2$  in the compactified moduli space of genus  $g$  curves. Indeed let us choose an almost complex structure on  $X$  compatible with the symplectic form. The restriction of the almost complex structure on each smooth fiber is integrable and so one obtains a map  $u$  of a punctured sphere into  $\mathcal{M}_g$ . By assumption the map  $f$  has a local complex model near the singular points, and it is easy to see that this gives an integrable almost complex structure in an entire neighborhood of the singular fibers. Thus we can extend  $u$  smoothly to a map of the closed sphere into the compactified moduli space. The isotopy class of this map is independent of the choices of almost complex structures on  $X$  and in the neighborhoods of the singular fibers.

Let  $\mathcal{L} \rightarrow \mathcal{M}_g$  be the Hodge line bundle and let  $c_1(\mathcal{L}) \in CH^1(\overline{\mathcal{M}}_g, \mathbb{Q})$  denote the natural extension of the first Chern class of  $\mathcal{L}$  to the Deligne-Mumford compactification of  $\mathcal{M}_g$ . We will need the following two preliminary lemmas:

**Lemma A.1** *For any symplectic Lefschetz fibration,*

$$\text{sign}(X) = 4\langle c_1(\mathcal{L}), [S] \rangle - \mu,$$

where  $\mu$  is the number of the singular fibers of  $f$  and  $\text{sign}(X)$  is the signature of  $X$ .

**Proof.** See [Smith98]. □

**Lemma A.2** *For a fibration with only separating vanishing cycles,  $\text{sign}(X) = -\mu$ .*

**Proof.** See [Ozbagci98]. □

As a consequence of the above two lemmas we get that for a symplectic Lefschetz fibration with monodromy group contained in the Torelli group:

$$\langle c_1(\mathcal{L}), [S] \rangle = 0.$$

Therefore Theorem 5.8 will be proven if we know the following:

**Lemma A.3** *Let  $X$  be a symplectic Lefschetz fibration with monodromy group contained in the Torelli group. Then:*

$$\langle c_1(\mathcal{L}), [S] \rangle > 0.$$

**Remark A.4** We need base  $S^2$  for this entire argument. There are fibrations over  $T^2$  with monodromy group in the Torelli group and no singular fibres.

**Proof** Let  $n$  denote the number of exceptional  $(-1)$ -spheres in the space  $X$  and let  $\varepsilon : X \rightarrow X_{\min}$  denote the contraction of all the  $(-1)$  spheres. Clearly since each new exceptional sphere contributes a homology class, we know that  $n$  is bounded above by the second Betti number  $b_2(X)$ . In fact without a loss of generality we may assume that  $n \leq b_2(X) - 2$ . Indeed the fibers of  $f : X \rightarrow S^2$

are not exceptional, and we can fiber sum  $X$  with itself to ensure that  $f : X \rightarrow S^2$  has a section with even square and in particular is not exceptional.

Furthermore we may assume that  $X_{\min}$  is not symplectomorphic to an irrational ruled surface. Indeed if  $\pi : X_{\min} \rightarrow C$  is a sphere bundle over a surface  $C$  of genus  $\geq 1$ , then  $H_1(C, \mathbb{Z}) = H_1(X_{\min}, \mathbb{Z}) = H_1(X, \mathbb{Z})$ . On the other hand since the geometric monodromy of  $f$  is contained in the Torelli group it follows that  $H_1(X_s, \mathbb{Z}) = H_1(X, \mathbb{Z})$  for all smooth fibers  $X_s \subset X$  of  $f$ . In particular  $g(C) = g(X_s) = g$  and so  $\pi|_{X_s} : X_s \rightarrow C$  must be a symplectomorphism. This shows that the natural map  $(\pi \circ \varepsilon, f) : X \rightarrow C \times S^2$  is a symplectic isomorphism in the complement of all  $(-1)$  spheres. This however implies that  $f$  is a trivial fibration which is a case we exclude.

Since  $X$  is a symplectic Lefschetz fibration of fiber genus  $g$  one has  $c_2(X) = 4 - 4g + \mu$ . Also observe that  $c_2(X) = \text{euler}(X) = 2 - 2b_1(X) + b_2(X)$  and  $b_1(X) = 2g$  since all vanishing cycles are by assumption null-homologous. So we get the estimate:

$$b_2(X) = 2 + \mu,$$

and hence

$$n \leq b_2(X) - 2 = \mu.$$

Now  $c_1^2(X) + c_2(X) = c_1^2(X_{\min}) - n + c_2(X)$  and using the above estimate on  $n$ , we see that

$$c_1^2(X) + c_2(X) > c_1^2(X_{\min}) + 2 - 4g \geq 2 - 4g - 2,$$

The last inequality follows from a powerful theorem of A.K.Liu [Liu96] asserting that for any *minimal* symplectic four-manifold  $Y$  which is not irrational ruled one has  $c_1^2(Y) \geq 0$ .

On the other hand the signature formula in Lemma A.1 gives

$$\begin{aligned} \frac{1}{12}[c_1^2(X) + c_2(X)] &= \frac{1}{4}[\text{sign}(X) + c_2(X)] \\ &= \frac{1}{4}[4\langle c_1(\mathcal{L}), [S] \rangle - \mu + 4(1 - g) + \mu] \\ &= \langle c_1(\mathcal{L}), [S] \rangle + 1 - g \end{aligned}$$

and hence

$$\langle c_1(\mathcal{L}), [S] \rangle > \frac{4g - 5}{6},$$

i.e.  $\langle c_1(\mathcal{L}), [S] \rangle > 0$  as long as  $g > 1$ . This proves the lemma since the case  $g = 1$  is clear.  $\square$

As a consequence of Theorem 5.8 we get

**Lemma A.5** *Let  $X$  be an arbitrary symplectic Lefschetz fibration. Then*

$$\text{sign}(X) + \mu > 0.$$

**Proof** As it follows from Theorem 5.8 there always exists a non homologous to zero vanishing cycle in  $X$ . By applying the local signature formula of [Ozbagci98] we see that the contribution of this singular fiber to  $\text{sign}(X)$  is either zero or one. The latter implies the lemma.  $\square$

As corollary we get:

**Corollary A.6** *Let  $X$  be an arbitrary symplectic Lefschetz fibration. Then:*

$$\langle c_1(\mathcal{L}), [S] \rangle > 0.$$

**Remark A.7** The above lemmas puts many restrictions on the possible monodromies for hyperelliptic families (see [Ozbagci98]).

**Remark A.8** The proof of Theorem 5.8 is related to a question of Gompf who asked whether  $c_2(X)$  is positive for symplectic four-manifolds which are not irrational ruled. In particular it is an interesting question if the minimal number of the singular fibers in a SLF is at least  $4(g-1)$  if  $X$  is not irrational ruled symplectic four-manifold.

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## References

- [Auroux98] D. Auroux, Géométrie approximativement holomorphe des variétés symplectiques compactes. Ecole Polytechnique Thesis 1998.
- [Birman88] J. Birman. Mapping class groups of surfaces. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 13–43. Amer. Math. Soc., Providence, RI, 1988.
- [Bogomolov-Katzarkov98] F. Bogomolov and L. Katzarkov. Symplectic fourmanifolds and projective surfaces. *Topology and its applications*, 88:79–110, 1998.
- [Cappell et al.94] S. Cappell, R. Lee, and E. Miller. On the Maslov index. *Comm. Pure Appl. Math.*, 47(2):121–186, 1994.
- [Donaldson98] S. Donaldson. Lefschetz fibrations in symplectic geometry. *Doc. Math. J. DMV.*, Extra Volume ICMII:309–314, 1998.
- [Friedman-Morgan94] R. Friedman and J. W. Morgan. Smooth four-manifolds and complex surfaces. Springer-Verlag, Berlin, 1994.
- [Gervais96] S. Gervais. Presentation and central extensions of mapping class groups. *Trans. Amer. Math. Soc.*, 348(8):3097–3132, 1996.
- [Gompf-Stipsicz] R. Gompf and A. Stipsicz. An introduction to 4-manifolds and Kirby calculus. book in preparation.
- [Gompf95] R. Gompf. A new construction of symplectic manifolds. *Ann. of Math. (2)*, 142(3):527–595, 1995.
- [Hain-Looijenga97] R. Hain and E. Looijenga. Mapping class groups and moduli spaces of curves. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 97–142. Amer. Math. Soc., Providence, RI, 1997.

- [Hain-Reed98] R. Hain and D. Reed. Geometric Proofs of Some Results of Morita. e-print: math.AG/9810054.
- [Harris84] J. Harris. Families of smooth curves. *Duke Math. J.*, 51(2):409–419, 1984.
- [Ivanov-McCarthy95] N. Ivanov and J.D. McCarthy. On injective homomorphisms between Teichmüller modular groups. preprint, available <http://www.math.msu.edu/~ivanov/i.ps>, 1995.
- [Kas80] A. Kas. On the handlebody decomposition associated to a Lefschetz fibration. *Pacific J. Math.*, 89(1):89–104, 1980.
- [Katzarkov et al.98] L. Katzarkov, T. Pantev, and S. Simpson. Non-abelian open orbit theorems. preprint, 1998.
- [Lion-Vergne80] G. Lion and M. Vergne. The Weil representation, Maslov index and theta series, volume 6 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1980.
- [Liu96] A.K. Liu. Some new applications of general wall crossing formula, Gompf’s conjecture and its applications, *Math. Res. Lett.*, 3(5):569–585, 1996.
- [McDuff-Salamon98] D. McDuff, D. Salamon. Introduction to Symplectic Topology, 2nd edition, Oxford University Press, 1998.
- [Moishezon77] B. Moishezon Complex surfaces and connected sums of complex projective planes, Lecture Notes in Mathematics, Vol. 603. Springer-Verlag, 1977.
- [Morita88] S. Morita. Characteristic classes of surface bundles and bounded cohomology. In *A fête of topology*, pages 233–257. Academic Press, Boston, MA, 1988.
- [Morita89] S. Morita. Families of Jacobian manifolds and characteristic classes of surface bundles. II. *Math. Proc. Cambridge Philos. Soc.*, 105(1):79–101, 1989.
- [Morita93] S. Morita. The structure of the mapping class group and characteristic classes of surface bundles. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 303–315. Amer. Math. Soc., Providence, RI, 1993.
- [Mumford65] D. Mumford. Picard groups of moduli problems. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 33–81. Harper & Row, New York, 1965.
- [Ozbagci98] B. Ozbagci. Signatures of Lefschetz fibrations. e-print: math.GT/9809178
- [Ozbagci-Stipsicz98] B. Ozbagci and A. Stipsicz. Noncomplex smooth 4-manifolds with genus-2 Lefschetz fibrations. e-print: math.GT/9810007
- [Powell78] J. Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.
- [SGA7II73] Groupes de monodromie en géométrie algébrique. II, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in Mathematics, Vol. 340.

- [Smitha] I. Smith. Lefschetz fibrations of genus two and sphere bundles over spheres. in preparation.
- [Smithb] I. Smith. Lefschetz fibrations of odd genus and torus bundles over tori. in preparation.
- [Smith98] I. Smith. PhD thesis, Oxford University, 1998.
- [Szpiro90] L. Szpiro. Discriminant et conducteur des courbes elliptiques. *Astérisque*, 183:7–18, 1990. Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988).
- [Wajnryb83] B. Wajnryb. A simple presentation for the mapping class group of an orientable surface. *Israel J. Math.*, 45(2-3):157–174, 1983.

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